Chapter 11

Radiation

Problem 11.1

From Eq. 11.17, $\mathbf{A} = -\frac{\mu_0 p_0 \omega}{4\pi}\frac{1}{r}\sin[\omega(t-r/c)](\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})$, so

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{A} &= -\frac{\mu_0 p_0\omega}{4\pi}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{1}{r}\sin[\omega(t-r/c)]\cos\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[-\sin^2\theta\frac{1}{r}\sin[\omega(t-r/c)]\right]\right\}\\
&= -\frac{\mu_0 p_0\omega}{4\pi}\left\{\frac{1}{r^2}\left(\sin[\omega(t-r/c)] - \frac{\omega r}{c}\cos[\omega(t-r/c)]\right)\cos\theta - \frac{2\sin\theta\cos\theta}{r^2\sin\theta}\sin[\omega(t-r/c)]\right\}\\
&= \mu_0\epsilon_0\left\{\frac{p_0\omega}{4\pi\epsilon_0}\left(\frac{1}{r^2}\sin[\omega(t-r/c)] + \frac{\omega}{rc}\cos[\omega(t-r/c)]\right)\cos\theta\right\}.
\end{aligned}$$

Meanwhile, from Eq. 11.12,

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{p_0\cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega^2}{c}\cos[\omega(t-r/c)] - \frac{\omega}{r}\sin[\omega(t-r/c)]\right\}\\
&= -\frac{p_0\omega}{4\pi\epsilon_0}\left\{\frac{1}{r^2}\sin[\omega(t-r/c)] + \frac{\omega}{rc}\cos[\omega(t-r/c)]\right\}\cos\theta. \quad \text{So } \boldsymbol{\nabla}\cdot\mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}. \quad \text{qed}
\end{aligned}$$

Problem 11.2

Eq. 11.14: $\boxed{V(\mathbf{r},t) = -\frac{\omega}{4\pi\epsilon_0 c}\frac{\mathbf{p}_0\cdot\hat{\mathbf{r}}}{r}\sin[\omega(t-r/c)].}$ Eq. 11.17: $\boxed{\mathbf{A}(\mathbf{r},t) = -\frac{\mu_0\omega}{4\pi}\frac{\mathbf{p}_0}{r}\sin[\omega(t-r/c)].}$

Now $\mathbf{p}_0\times\hat{\mathbf{r}} = p_0\sin\theta\,\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{r}}\times(\mathbf{p}_0\times\hat{\mathbf{r}}) = p_0\sin\theta(\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}}) = -p_0\sin\theta\,\hat{\boldsymbol{\theta}}$, so

Eq. 11.18: $\boxed{\mathbf{E}(\mathbf{r},t) = \frac{\mu_0\omega^2}{4\pi}\frac{\hat{\mathbf{r}}\times(\mathbf{p}_0\times\hat{\mathbf{r}})}{r}\cos[\omega(t-r/c)].}$ Eq. 11.19: $\boxed{\mathbf{B}(\mathbf{r},t) = -\frac{\mu_0\omega^2}{4\pi c}\frac{(\mathbf{p}_0\times\hat{\mathbf{r}})}{r}\cos[\omega(t-r/c)].}$

Eq. 11.21: $\boxed{\langle\mathbf{S}\rangle = \frac{\mu_0\omega^4}{32\pi^2 c}\frac{(\mathbf{p}_0\times\hat{\mathbf{r}})^2}{r^2}\hat{\mathbf{r}}.}$

Problem 11.3

$P = I^2R = q_0^2\omega^2\sin^2(\omega t)R$ (Eq. 11.15) $\Rightarrow \langle P\rangle = \frac{1}{2}q_0^2\omega^2 R$. Equate this to Eq. 11.22:

$\frac{1}{2}q_0^2\omega^2 R = \frac{\mu_0 q_0^2 d^2\omega^4}{12\pi c} \Rightarrow \boxed{R = \frac{\mu_0 d^2\omega^2}{6\pi c};}$ or, since $\omega = \frac{2\pi c}{\lambda}$,

$$R = \frac{\mu_0 d^2}{6\pi c}\frac{4\pi^2c^2}{\lambda^2} = \frac{2}{3}\pi\mu_0 c\left(\frac{d}{\lambda}\right)^2 = \frac{2}{3}\pi(4\pi\times10^{-7})(3\times10^8)\left(\frac{d}{\lambda}\right)^2 = 80\pi^2\left(\frac{d}{\lambda}\right)^2\,\Omega = \boxed{789.6(d/\lambda)^2\,\Omega.}$$

For the wires in an ordinary radio, with $d = 5\times10^{-2}$ m and (say) $\lambda = 10^3$ m, $R = 790(5\times10^{-5})^2 = 2\times10^{-6}\,\Omega$, which is negligible compared to the Ohmic resistance.

Problem 11.4

By the superposition principle, we can *add* the potentials of the two dipoles. Let's first express V (Eq. 11.14) in Cartesian coordinates: $V(x,y,z,t) = -\dfrac{p_0\omega}{4\pi\epsilon_0 c}\left(\dfrac{z}{x^2+y^2+z^2}\right)\sin[\omega(t-r/c)]$. That's for an oscillating dipole along the z axis. For one along x or y, we just change z to x or y. In the present case, $\mathbf{p} = p_0[\cos(\omega t)\,\hat{\mathbf{x}} + \cos(\omega t - \pi/2)\,\hat{\mathbf{y}}]$, so the one along y is delayed by a phase angle $\pi/2$: $\sin[\omega(t-r/c)] \to \sin[\omega(t-r/c)-\pi/2] = -\cos[\omega(t-r/c)]$ (just let $\omega t \to \omega t - \pi/2$). Thus

$$\begin{aligned} V &= -\frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\frac{x}{x^2+y^2+z^2}\sin[\omega(t-r/c)] - \frac{y}{x^2+y^2+z^2}\cos[\omega(t-r/c)]\right\} \\ &= \boxed{-\frac{p_0\omega}{4\pi\epsilon_0 c}\frac{\sin\theta}{r}\{\cos\phi\sin[\omega(t-r/c)] - \sin\phi\cos[\omega(t-r/c)]\}.} \quad \text{Similarly,} \\ \mathbf{A} &= \boxed{-\frac{\mu_0 p_0\omega}{4\pi r}\{\sin[\omega(t-r/c)]\,\hat{\mathbf{x}} - \cos[\omega(t-r/c)]\,\hat{\mathbf{y}}\}.} \end{aligned}$$

We *could* get the fields by differentiating these potentials, but I prefer to work with Eqs. 11.18 and 11.19, using superposition. Since $\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$, and $\cos\theta = z/r$, Eq. 11.18 can be written $\mathbf{E} = \dfrac{\mu_0 p_0\omega^2}{4\pi r}\cos[\omega(t-r/c)]\left(\hat{\mathbf{z}} - \dfrac{z}{r}\hat{\mathbf{r}}\right)$. In the case of the rotating dipole, therefore,

$$\begin{aligned} \mathbf{E} &= \boxed{\frac{\mu_0 p_0\omega^2}{4\pi r}\left\{\cos[\omega(t-r/c)]\left(\hat{\mathbf{x}} - \frac{x}{r}\hat{\mathbf{r}}\right) + \sin[\omega(t-r/c)]\left(\hat{\mathbf{y}} - \frac{y}{r}\hat{\mathbf{r}}\right)\right\},} \\ \mathbf{B} &= \boxed{\frac{1}{c}(\hat{\mathbf{r}}\times\mathbf{E}).} \end{aligned}$$

$\mathbf{S} = \dfrac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \dfrac{1}{\mu_0 c}[\mathbf{E}\times(\hat{\mathbf{r}}\times\mathbf{E})] = \dfrac{1}{\mu_0 c}\left[E^2\,\hat{\mathbf{r}} - (\mathbf{E}\cdot\hat{\mathbf{r}})\mathbf{E}\right] = \dfrac{E^2}{\mu_0 c}\hat{\mathbf{r}}$ (notice that $\mathbf{E}\cdot\hat{\mathbf{r}} = 0$). Now

$$E^2 = \left(\frac{\mu_0 p_0\omega^2}{4\pi r}\right)^2\left\{a^2\cos^2[\omega(t-r/c)] + b^2\sin^2[\omega(t-r/c)] + 2(\mathbf{a}\cdot\mathbf{b})\sin[\omega(t-r/c)]\cos[\omega(t-r/c)]\right\},$$

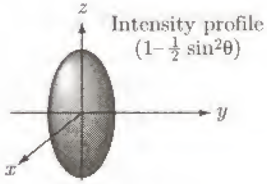
where $\mathbf{a} \equiv \hat{\mathbf{x}} - (x/r)\hat{\mathbf{r}}$ and $\mathbf{b} \equiv \hat{\mathbf{y}} - (y/r)\hat{\mathbf{r}}$. Noting that $\hat{\mathbf{x}}\cdot\mathbf{r} = x$ and $\hat{\mathbf{y}}\cdot\mathbf{r} = y$, we have

$$a^2 = 1 + \frac{x^2}{r^2} - 2\frac{x^2}{r^2} = 1 - \frac{x^2}{r^2};\ b^2 = 1 - \frac{y^2}{r^2};\ \mathbf{a}\cdot\mathbf{b} = -\frac{y}{r}\frac{x}{r} - \frac{x}{r}\frac{y}{r} + \frac{xy}{r^2} = -\frac{xy}{r^2}.$$

$$\begin{aligned}
E^2 &= \left(\frac{\mu_0 p_0 \omega^2}{4\pi r}\right)^2 \left\{\left(1-\frac{x^2}{r^2}\right)\cos^2[\omega(t-r/c)] + \left(1-\frac{y^2}{r^2}\right)\sin^2[\omega(t-r/c)]\right.\\
&\quad \left. - 2\frac{xy}{r^2}\sin[\omega(t-r/c)]\cos[\omega(t-r/c)]\right\}\\
&= \left(\frac{\mu_0 p_0 \omega^2}{4\pi r}\right)^2 \left\{1 - \frac{1}{r^2}\left(x^2\cos^2[\omega(t-r/c)] + 2xy\sin[\omega(t-r/c)]\cos[\omega(t-r/c)] + y^2\sin^2[\omega(t-r/c)]\right)\right\}\\
&= \left(\frac{\mu_0 p_0 \omega^2}{4\pi r}\right)^2 \left\{1 - \frac{1}{r^2}\left(x\cos[\omega(t-r/c)] + y\sin[\omega(t-r/c)]\right)^2\right\}
\end{aligned}$$

But $x = r\sin\theta\cos\phi$ and $y = r\sin\theta\sin\phi$.

$$\begin{aligned}
&= \left(\frac{\mu_0 p_0 \omega^2}{4\pi r}\right)^2 \left\{1 - \sin^2\theta\left(\cos\phi\cos[\omega(t-r/c)] + \sin\phi\sin[\omega(t-r/c)]\right)^2\right\}\\
&= \left(\frac{\mu_0 p_0 \omega^2}{4\pi r}\right)^2 \left\{1 - \left(\sin\theta\cos[\omega(t-r/c)-\phi]\right)^2\right\}.
\end{aligned}$$



$$\mathbf{S} = \boxed{\frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2 \left\{1 - \left(\sin\theta\cos[\omega(t-r/c)-\phi]\right)^2\right\}\hat{\mathbf{r}}.}$$

$$\langle\mathbf{S}\rangle = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2 \left[1 - \frac{1}{2}\sin^2\theta\right]\hat{\mathbf{r}}.$$

$$\begin{aligned}
P &= \int \langle\mathbf{S}\rangle\cdot d\mathbf{a} = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi}\right)^2 \int \frac{1}{r^2}\left(1-\frac{1}{2}\sin^2\theta\right) r^2\sin\theta\, d\theta\, d\phi\\
&= \frac{\mu_0 p_0^2\omega^4}{16\pi^2 c}2\pi\left[\int_0^\pi \sin\theta\, d\theta - \frac{1}{2}\int_0^\pi \sin^3\theta\, d\theta\right] = \frac{\mu_0 p_0^2\omega^4}{8\pi c}\left(2 - \frac{1}{2}\cdot\frac{4}{3}\right) = \boxed{\frac{\mu_0 p_0^2\omega^4}{6\pi c}.}
\end{aligned}$$

This is *twice* the power radiated by either oscillating dipole alone (Eq. 11.22). In general, $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0}\left[(\mathbf{E}_1+\mathbf{E}_2)\times(\mathbf{B}_1+\mathbf{B}_2)\right] = \frac{1}{\mu_0}\left[(\mathbf{E}_1\times\mathbf{B}_1) + (\mathbf{E}_2\times\mathbf{B}_2) + (\mathbf{E}_1\times\mathbf{B}_2) + (\mathbf{E}_2\times\mathbf{B}_1)\right] = \mathbf{S}_1 + \mathbf{S}_2 +$ cross terms. In this particular case, the fields of 1 and 2 are 90° out of phase, so the cross terms go to zero in the time averaging, and the total power radiated is just the sum of the two individual powers.

Problem 11.5

Go back to Eq. 11.33:

$$\mathbf{A} = \frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}\cos[\omega(t-r/c)] - \frac{\omega}{c}\sin[\omega(t-r/c)]\right\}\hat{\boldsymbol{\phi}}.$$

Since $V = 0$ here,

$$\begin{aligned}
\mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}(-\omega)\sin[\omega(t-r/c)] - \frac{\omega}{c}\omega\cos[\omega(t-r/c)]\right\}\hat{\boldsymbol{\phi}} \\
&= \boxed{\frac{\mu_0 m_0 \omega}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}\sin[\omega(t-r/c)] + \frac{\omega}{c}\cos[\omega(t-r/c)]\right\}\hat{\boldsymbol{\phi}}.} \\
\mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right)\hat{\boldsymbol{\theta}} \\
&= \frac{\mu_0 m_0}{4\pi}\left\{\frac{1}{r\sin\theta}\frac{2\sin\theta\cos\theta}{r}\left[\frac{1}{r}\cos[\omega(t-r/c)] - \frac{\omega}{c}\sin[\omega(t-r/c)]\right]\hat{\mathbf{r}}\right. \\
&\qquad \left. - \frac{\sin\theta}{r}\left[-\frac{1}{r^2}\cos[\omega(t-r/c)] + \frac{\omega}{rc}\sin[\omega(t-r/c)] - \frac{\omega}{c}\left(-\frac{\omega}{c}\right)\cos[\omega(t-r/c)]\right]\hat{\boldsymbol{\theta}}\right\} \\
&= \boxed{\begin{aligned}\frac{\mu_0 m_0}{4\pi} & \left\{\frac{2\cos\theta}{r^2}\left[\frac{1}{r}\cos[\omega(t-r/c)] - \frac{\omega}{c}\sin[\omega(t-r/c)]\right]\hat{\mathbf{r}}\right. \\
& \left. -\frac{\sin\theta}{r}\left[-\frac{1}{r^2}\cos[\omega(t-r/c)] + \frac{\omega}{rc}\sin[\omega(t-r/c)] + \left(\frac{\omega}{c}\right)^2\cos[\omega(t-r/c)]\right]\hat{\boldsymbol{\theta}}\right\}.\end{aligned}}
\end{aligned}$$

These are precisely the fields we studied in Prob. 9.33, with $A \to \dfrac{\mu_0 m_0 \omega^2}{4\pi c}$. The Poynting vector (quoting the solution to that problem) is

$$\boxed{\begin{aligned}\mathbf{S} = \frac{\mu_0 m_0^2 \omega^3}{16\pi^2 c^2}\left(\frac{\sin\theta}{r^2}\right) & \left\{\frac{2\cos\theta}{r}\left[\left(1-\frac{c^2}{\omega^2 r^2}\right)\sin u\cos u + \frac{c}{\omega r}\left(\cos^2 u - \sin^2 u\right)\right]\hat{\boldsymbol{\theta}}\right. \\
& \left.\sin\theta\left[\left(-\frac{2}{r}+\frac{c^2}{\omega^2 r^3}\right)\sin u\cos u + \frac{\omega}{c}\cos^2 u + \frac{c}{\omega r^2}\left(\sin^2 u - \cos^2 u\right)\right]\hat{\mathbf{r}}\right\},\end{aligned}}$$

where $u \equiv -\omega(t-r/c)$. The intensity is $\boxed{\langle\mathbf{S}\rangle = \dfrac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\dfrac{\sin^2\theta}{r^2}\hat{\mathbf{r}},}$ the same as Eq. 11.39.

Problem 11.6

$$I^2R = I_0^2 R\cos^2(\omega t) \Rightarrow \langle P\rangle = \frac{1}{2}I_0^2 R = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3} = \frac{\mu_0 \pi^2 b^4 I_0^2 \omega^4}{12\pi c^3}, \text{ so } \boxed{R = \frac{\mu_0 \pi b^4 \omega^4}{6c^3};} \text{ or, since } \omega = \frac{2\pi c}{\lambda},$$

$$R = \frac{\mu_0 \pi b^4}{6c^3}\frac{16\pi^4 c^4}{\lambda^4} = \boxed{\frac{8}{3}\pi^5 \mu_0 c\left(\frac{b}{\lambda}\right)^4} = \frac{8}{3}(\pi^5)(4\pi\times10^{-7})(3\times10^8)(b/\lambda)^4 = \boxed{3.08\times10^5 (b/\lambda)^4\,\Omega.}$$

Because $b \ll \lambda$, and R goes like the *fourth* power of this small number, R is typically much smaller than the electric radiative resistance (Prob. 11.3). For the dimensions we used in Prob. 11.3 ($b = 5\,\text{cm}$ and $\lambda = 10^3\,\text{m}$), $R = 3\times10^5(5\times10^{-5})^4 = 2\times10^{-12}\,\Omega$, which is a millionth of the comparable electrical radiative resistance.

Problem 11.7

With $\alpha = 90°$, Eq. 7.68 $\Rightarrow \mathbf{E}' = c\mathbf{B}$, $\mathbf{B}' = -\mathbf{E}/c$, $q'_m = -cq_e \Rightarrow m_0 \equiv q'_m d = -cq_e d = -cp_0$. So

$$\begin{aligned}
\mathbf{E}' &= c\left\{-\frac{\mu_0(-m_0/c)\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\boldsymbol{\phi}}\right\} = \boxed{\frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\boldsymbol{\phi}}.} \\
\mathbf{B}' &= -\frac{1}{c}\left\{-\frac{\mu_0(-m_0/c)\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\boldsymbol{\theta}}\right\} = \boxed{-\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\boldsymbol{\theta}}.}
\end{aligned}$$

These are *identical* to the fields of an Ampére dipole (Eqs. 11.36 and 11.37), which is consistent with our general experience that the two models generate identical fields *except right at* the dipole (not relevant here, since we're in the radiation zone).

Problem 11.8

$\mathbf{p}(t) = p_0[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}] \Rightarrow \ddot{\mathbf{p}}(t) = -\omega^2 p_0[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}] \Rightarrow$

$[\ddot{\mathbf{p}}(t)]^2 = \omega^4 p_0^2[\cos^2(\omega t) + \sin^2(\omega t)] = p_0^2\omega^4$. So Eq. 11.59 says $\boxed{\mathbf{S} = \frac{\mu_0 p_0^2\omega^4}{16\pi^2 c}\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}}.}$ (This appears to disagree with the answer to Prob. 11.4. The reason is that in Eq. 11.59 the polar axis is along the direction of $\ddot{\mathbf{p}}(t_0)$; as the dipole rotates, so do the axes. Thus the angle θ here is not the same as in Prob. 11.4.) Meanwhile, Eq. 11.60 says $\boxed{P = \frac{\mu_0 p_0^2\omega^4}{6\pi c}.}$ (This *does* agree with Prob. 11.4, because we have now integrated over all angles, and the orientation of the polar axis irrelevant.)

Problem 11.9

At $t = 0$ the dipole moment of the ring is

$$\begin{aligned}\mathbf{p}_0 &= \int \lambda\mathbf{r}\,dl = \int(\lambda_0\sin\phi)(b\sin\phi\,\hat{\mathbf{y}} + b\cos\phi\,\hat{\mathbf{x}})b\,d\phi = \lambda_0 b^2\left(\hat{\mathbf{y}}\int_0^{2\pi}\sin^2\phi\,d\phi + \hat{\mathbf{x}}\int_0^{2\pi}\sin\phi\cos\phi\,d\phi\right)\\ &= \lambda b^2(\pi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{x}}) = \pi b^2\lambda_0\,\hat{\mathbf{y}}.\end{aligned}$$

As it rotates (counterclockwise, say) $\mathbf{p}(t) = p_0[\cos(\omega t)\,\hat{\mathbf{y}} - \sin(\omega t)\,\hat{\mathbf{x}}]$, so $\ddot{\mathbf{p}} = -\omega^2\mathbf{p}$, and hence $(\ddot{\mathbf{p}})^2 = \omega^4 p_0^2$. Therefore (Eq. 11.60) $P = \frac{\mu_0}{6\pi c}\omega^4(\pi b^2\lambda_0)^2 = \boxed{\frac{\pi\mu_0\omega^4 b^4\lambda_0^2}{6c}.}$

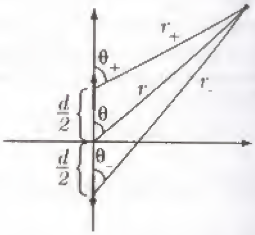
Problem 11.10

$\mathbf{p} = -ey\,\hat{\mathbf{y}}$, $y = \frac{1}{2}gt^2$, so $\mathbf{p} = -\frac{1}{2}get^2\,\hat{\mathbf{y}}$; $\ddot{\mathbf{p}} = -ge\,\hat{\mathbf{y}}$. Therefore (Eq. 11.60) : $P = \frac{\mu_0}{6\pi c}(ge)^2$. Now, the time it takes to fall a distance h is given by $h = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{2h/g}$, so the energy radiated in falling a distance h is $U_{\text{rad}} = Pt = \frac{\mu_0(ge)^2}{6\pi c}\sqrt{2h/g}$. Meanwhile, the potential energy lost is $U_{\text{pot}} = mgh$. So the *fraction* is

$$f = \frac{U_{\text{rad}}}{U_{\text{pot}}} = \frac{\mu_0 g^2 e^2}{6\pi c}\sqrt{\frac{2h}{g}}\frac{1}{mgh} = \boxed{\frac{\mu_0 e^2}{6\pi mc}\sqrt{\frac{2g}{h}}.} = \frac{(4\pi\times10^{-7})(1.6\times10^{-19})^2}{6\pi(9.11\times10^{-31})(3\times10^8)}\sqrt{\frac{(2)(9.8)}{(0.02)}} = \boxed{2.76\times10^{-22}.}$$

Evidently *almost* all the energy goes into kinetic form (as indeed I *assumed* in saying $y = \frac{1}{2}gt^2$).

Problem 11.11



(a)
$$\begin{aligned}
V_\pm &= \mp\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta_\pm}{r_\pm}\right)\sin[\omega(t-r_\pm/c)]. \quad V_{\text{tot}} = V_+ + V_-.\\
r_\pm &= \sqrt{r^2+(d/2)^2\mp 2r(d/2)\cos\theta} \cong r\sqrt{1\mp(d/r)\cos\theta} \cong r\left(1\mp\frac{d}{2r}\cos\theta\right).\\
\frac{1}{r_\pm} &\cong \frac{1}{r}\left(1\pm\frac{d}{2r}\cos\theta\right).
\end{aligned}$$

$$\begin{aligned}
\cos\theta_\pm &= \frac{r\cos\theta\mp(d/2)}{r_\pm} = r\left(\cos\theta\mp\frac{d}{2r}\right)\frac{1}{r}\left(1\pm\frac{d}{2r}\cos\theta\right) = \cos\theta\pm\frac{d}{2r}\cos^2\theta\mp\frac{d}{2r}\\
&= \cos\theta\mp\frac{d}{2r}(1-\cos^2\theta) = \cos\theta\mp\frac{d}{2r}\sin^2\theta.\\
\sin[\omega(t-r_\pm/c)] &= \sin\left\{\omega\left[t-\frac{r}{c}\left(1\mp\frac{d}{2r}\cos\theta\right)\right]\right\} = \sin\left(\omega t_0\pm\frac{\omega d}{2c}\cos\theta\right), \text{ where } t_0\equiv t-r/c.\\
&= \sin(\omega t_0)\cos\left(\frac{\omega d}{2c}\cos\theta\right)\pm\cos(\omega t_0)\sin\left(\frac{\omega d}{2c}\cos\theta\right)\cong\sin(\omega t_0)\pm\frac{\omega d}{2c}\cos\theta\cos(\omega t_0).
\end{aligned}$$

$$\begin{aligned}
V_\pm &= \mp\frac{p_0\omega}{4\pi\epsilon_0 cr}\left\{\left(1\pm\frac{d}{2r}\cos\theta\right)\left(\cos\theta\mp\frac{d}{2r}\sin^2\theta\right)\left[\sin(\omega t_0)\pm\frac{\omega d}{2c}\cos\theta\cos(\omega t_0)\right]\right\}\\
&= \mp\frac{p_0\omega}{4\pi\epsilon_0 cr}\left\{\left(\cos\theta\mp\frac{d}{2r}\sin^2\theta\pm\frac{d}{2r}\cos^2\theta\right)\left[\sin(\omega t_0)\pm\frac{\omega d}{2c}\cos\theta\cos(\omega t_0)\right]\right\}\\
&= \mp\frac{p_0\omega}{4\pi\epsilon_0 cr}\left[\cos\theta\sin(\omega t_0)\pm\frac{\omega d}{2c}\cos^2\theta\cos(\omega t_0)\pm\frac{d}{2r}\left(\cos^2\theta-\sin^2\theta\right)\sin(\omega t_0\right].\\
V_{\text{tot}} &= -\frac{p_0\omega}{4\pi\epsilon_0 cr}\left[\frac{\omega d}{c}\cos^2\theta\cos(\omega t_0)+\frac{d}{r}\left(\cos^2\theta-\sin^2\theta\right)\sin(\omega t_0)\right]\\
&= \boxed{-\frac{p_0\omega^2 d}{4\pi\epsilon_0 c^2 r}\left[\cos^2\theta\cos(\omega t_0)+\frac{c}{\omega r}\left(\cos^2\theta-\sin^2\theta\right)\sin(\omega t_0\right].}
\end{aligned}$$

In the radiation zone ($r \gg \omega/c$) the second term is negligible, so $\boxed{V = -\frac{p_0\omega^2 d}{4\pi\epsilon_0 c^2 r}\cos^2\theta\cos[\omega(t-r/c)].}$

Meanwhile

$$\begin{aligned}
\mathbf{A}_\pm &= \mp\frac{\mu_0 p_0\omega}{4\pi r_\pm}\sin[\omega(t-r_\pm/c)]\,\hat{\mathbf{z}}\\
&= \mp\frac{\mu_0 p_0\omega}{4\pi r}\left\{\left(1\pm\frac{d}{2r}\cos\theta\right)\left[\sin(\omega t_0)\pm\frac{\omega d}{2c}\cos\theta\cos(\omega t_0)\right]\right\}\hat{\mathbf{z}}\\
&= \mp\frac{\mu_0 p_0\omega}{4\pi r}\left[\sin(\omega t_0)\pm\frac{\omega d}{2c}\cos\theta\cos(\omega t_0)\pm\frac{d}{2r}\cos\theta\sin(\omega t_0)\right]\hat{\mathbf{z}}.\\
\mathbf{A}_{\text{tot}} &= \mathbf{A}_+ + \mathbf{A}_= -\frac{\mu_0 p_0\omega}{4\pi r}\left[\frac{\omega d}{c}\cos\theta\cos(\omega t_0)+\frac{d}{r}\cos\theta\sin(\omega t_0)\right]\hat{\mathbf{z}}\\
&= \boxed{-\frac{\mu_0 p_0\omega^2 d}{4\pi c r}\cos\theta\left[\cos(\omega t_0)+\frac{c}{\omega r}\sin(\omega t_0)\right]\hat{\mathbf{z}}.}
\end{aligned}$$

In the radiation zone, $\boxed{\mathbf{A} = -\dfrac{\mu_0 p_0 \omega^2 d}{4\pi c r} \cos\theta \cos[\omega(t - r/c)]\,\hat{\mathbf{z}}.}$

(b) To simplify the notation, let $\alpha \equiv -\dfrac{\mu_0 p_0 \omega^2 d}{4\pi}$. Then

$$\begin{aligned}
V &= \alpha \frac{\cos^2\theta}{r} \cos[\omega(t - r/c)];\\
\boldsymbol{\nabla} V &= \frac{\partial V}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\hat{\boldsymbol{\theta}} = \alpha \cos^2\theta \left\{ -\frac{1}{r^2}\cos[\omega(t - r/c)] + \frac{\omega}{rc}\sin[\omega(t - r/c)] \right\} \hat{\mathbf{r}}\\
&\quad + \alpha \frac{-2\cos\theta\sin\theta}{r^2} \cos[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}} = \alpha \frac{\omega}{c}\frac{\cos^2\theta}{r} \sin[\omega(t - r/c)]\,\hat{\mathbf{r}} \quad \text{(in the radiation zone).}\\
\mathbf{A} &= \frac{\alpha}{c}\frac{\cos\theta}{r} \cos[\omega(t - r/c)] \left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \right). \quad \frac{\partial \mathbf{A}}{\partial t} = -\frac{\alpha\omega}{c}\frac{\cos\theta}{r} \sin[\omega(t - r/c)] \left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \right).
\end{aligned}$$

$$\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\alpha\omega}{cr} \sin[\omega(t - r/c)] \left(\cos^2\theta\,\hat{\mathbf{r}} - \cos^2\theta\,\hat{\mathbf{r}} + \sin\theta\cos\theta\,\hat{\boldsymbol{\theta}} \right)\\
&= \boxed{-\frac{\alpha\omega}{cr} \sin\theta\cos\theta \sin[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}}.}\\
\mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}}\\
&= \frac{\alpha}{cr} \left\{ \frac{\partial}{\partial r}\left(\cos\theta \cos[\omega(t - r/c)](-\sin\theta) \right) - \frac{\partial}{\partial \theta}\left[\frac{\cos^2\theta}{r} \cos[\omega(t - r/c)] \right] \right\} \hat{\boldsymbol{\phi}}\\
&= \frac{\alpha}{cr}(-\sin\theta\cos\theta)\frac{\omega}{c} \sin[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}} \text{ (in the radiation zone)} = \boxed{-\frac{\alpha\omega}{c^2 r} \sin\theta\cos\theta \sin[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}}.}
\end{aligned}$$

Notice that $\mathbf{B} = \dfrac{1}{c}(\hat{\mathbf{r}} \times \mathbf{E})$ and $\mathbf{E} \cdot \hat{\mathbf{r}} = 0$.

$$\begin{aligned}
\mathbf{S} &= \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{1}{\mu_0 c}\mathbf{E} \times (\hat{\mathbf{r}} \times \mathbf{E}) = \frac{1}{\mu_0 c}\left[E^2\,\hat{\mathbf{r}} - (\mathbf{E} \cdot \hat{\mathbf{r}})\mathbf{E} \right] = \frac{E^2}{\mu_0 c}\,\hat{\mathbf{r}}\\
&= \boxed{\frac{1}{\mu_0 c}\left\{ \frac{\alpha\omega}{rc} \sin\theta\cos\theta \sin[\omega(t - r/c)] \right\}^2 \hat{\mathbf{r}}.} \quad \boxed{I = \frac{1}{2\mu_0 c}\left(\frac{\alpha\omega}{rc} \sin\theta\cos\theta \right)^2.}\\
P &= \int \langle \mathbf{S} \rangle \cdot d\mathbf{a} = \frac{1}{\mu_0 c}\left(\frac{\alpha\omega}{c} \right)^2 \int \sin^2\theta\cos^2\theta \sin\theta\, d\theta\, d\phi = \frac{1}{2\mu_0 c}\left(\frac{\alpha\omega}{c} \right)^2 2\pi \int_0^\pi (1 - \cos^2\theta)\cos^2\theta \sin\theta\, d\theta.\\
&\quad \text{The integral is : } -\left.\frac{\cos^3\theta}{3}\right|_0^\pi + \left.\frac{\cos^5\theta}{5}\right|_0^\pi = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}.\\
&= \frac{1}{2\mu_0 c}\frac{\omega^2}{c^2}\frac{\mu_0^2}{16\pi^2}(p_0 d)^2 \omega^4 2\pi \frac{4}{15} = \boxed{\frac{\mu_0}{60\pi c^3}(p_0 d)^2 \omega^6.}
\end{aligned}$$

Notice that it goes like ω^6, whereas *dipole* radiation goes like ω^4.

Problem 11.12

Here $V = 0$ (since the ring is neutral), and the current depends only on t (not on position), so the retarded vector potential (Eq. 11.52) is $\mathbf{A}(\mathbf{r}, t) = \dfrac{\mu_0}{4\pi} \oint \dfrac{I(t - \mathscr{r}/c)}{\mathscr{r}}\, d\mathbf{l}'$. But in this case it does *not* suffice to replace $\mathscr{r}$ by

r in the denominator—that would lead to Eq. 11.54, and hence to $\mathbf{A} = 0$ (since $\mathbf{p} = 0$). Instead, use Eq. 11.30: $\dfrac{1}{\mathscr{r}} \cong \dfrac{1}{r}\left(1 + \dfrac{b}{r}\sin\theta\cos\phi'\right)$. Meanwhile, $d\mathbf{l}' = b\,d\phi'\,\hat{\boldsymbol{\phi}} = b(-\sin\phi'\,\hat{\mathbf{x}} + \cos\phi'\,\hat{\mathbf{y}})\,d\phi'$, and

$$I(t - \mathscr{r}/c) \cong I(t - r/c + (b/c)\sin\theta\cos\phi') = I(t_0 + (b/c)\sin\theta\cos\phi') \cong I(t_0) + \dot{I}(t_0)\frac{b}{c}\sin\theta\cos\phi'$$

(carrying all terms to first order in b). As always, $t_0 = t - r/c$. (From now on I'll suppress the argument: I, $\dot{I}$, etc. are all to be evaluated at t_0.) Then

$$\begin{aligned}
\mathbf{A}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\oint \frac{1}{r}\left(1 + \frac{b}{r}\sin\theta\cos\phi'\right)\left(I + \dot{I}\frac{b}{c}\sin\theta\cos\phi'\right) b(-\sin\phi'\,\hat{\mathbf{x}} + \cos\phi'\,\hat{\mathbf{y}})\,d\phi' \\
&\cong \frac{\mu_0 b}{4\pi r}\int_0^{2\pi}\left[I + \dot{I}\frac{b}{c}\sin\theta\cos\phi' + I\frac{b}{r}\sin\theta\cos\phi'\right](-\sin\phi'\,\hat{\mathbf{x}} + \cos\phi'\,\hat{\mathbf{y}})\,d\phi'.
\end{aligned}$$

But $\displaystyle\int_0^{2\pi}\sin\phi'\,d\phi' = \int_0^{2\pi}\cos\phi'\,d\phi' = \int_0^{2\pi}\sin\phi'\cos\phi'\,d\phi' = 0$, while $\displaystyle\int_0^{2\pi}\cos^2\phi'\,d\phi' = \pi$.

$$= \frac{\mu_0 b}{4\pi r}(\pi\,\hat{\mathbf{y}})\left[\dot{I}\frac{b}{c}\sin\theta + I\frac{b}{r}\sin\theta\right] = \frac{\mu_0 b^2}{4r^2}\sin\theta\left(I + \frac{r}{c}\dot{I}\right)\hat{\mathbf{y}}.$$

In general (i.e. for points *not* on the $x\,z$ plane) $\hat{\mathbf{y}} \to \hat{\boldsymbol{\phi}}$; moreover, in the radiation zone we are not interested in terms that go like $1/r^2$, so $\boxed{\mathbf{A}(\mathbf{r},t) = \dfrac{\mu_0 b^2}{4c}\left[\dot{I}(t - r/c)\right]\dfrac{\sin\theta}{r}\,\hat{\boldsymbol{\phi}}.}$

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= -\frac{\partial\mathbf{A}}{\partial t} = \boxed{-\frac{\mu_0 b^2}{4c}\left[\ddot{I}(t - r/c)\right]\frac{\sin\theta}{r}\,\hat{\boldsymbol{\phi}}.} \\
\mathbf{B}(\mathbf{r},t) &= \boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta)\,\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\,\hat{\boldsymbol{\theta}} \\
&= \frac{\mu_0 b^2}{4c}\left[\frac{\dot{I}}{r\sin\theta}\frac{1}{r}2\sin\theta\cos\theta\,\hat{\mathbf{r}} - \frac{1}{r}\ddot{I}\left(-\frac{1}{c}\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right] = \boxed{\frac{\mu_0 b^2}{4c^2}\ddot{I}\frac{\sin\theta}{r}\,\hat{\boldsymbol{\theta}}.} \\
\mathbf{S} &= \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0 c}\left(\frac{\mu_0 b^2}{4c}\ddot{I}\frac{\sin\theta}{r}\right)^2\left(-\hat{\boldsymbol{\phi}}\times\hat{\boldsymbol{\theta}}\right) = \boxed{\frac{\mu_0}{16c^3}\left(b^2\ddot{I}\right)^2\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}}.} \\
P &= \int\mathbf{S}\cdot d\mathbf{a} = \frac{\mu_0}{16c^3}\left(b^2\ddot{I}\right)^2\int\frac{\sin^2\theta}{r^2}r^2\sin\theta\,d\theta\,d\phi = \frac{\mu_0}{16c^3}\left(b^2\ddot{I}\right)^2(2\pi)\left(\frac{4}{3}\right) = \frac{\mu_0\pi}{6c^3}\left(b^2\ddot{I}\right)^2 \\
&= \boxed{\frac{\mu_0\ddot{m}^2}{6\pi c^3}.} \quad \text{(Note that } m = I\pi b^2\text{, so } \ddot{m} = \ddot{I}\pi b^2\text{.)}
\end{aligned}$$

Problem 11.13

(a) $P = \dfrac{\mu_0 q^2 a^2}{6\pi c}$, and the time it takes to come to rest is $t = v_0/a$, so the energy radiated is $U_{\text{rad}} = Pt = \dfrac{\mu_0 q^2 a^2}{6\pi c}\dfrac{v_0}{a}$. The initial kinetic energy was $U_{\text{kin}} = \frac{1}{2}mv_0^2$, so the fraction radiated is $f = \dfrac{U_{\text{rad}}}{U_{\text{kin}}} = \boxed{\dfrac{\mu_0 q^2 a}{3\pi m v_0 c}.}$

(b) $d = \dfrac{1}{2}at^2 = \dfrac{1}{2}a\dfrac{v_0^2}{a^2} = \dfrac{v_0^2}{2a}$, so $a = \dfrac{v_0^2}{2d}$. Then

$$f = \frac{\mu_0 q^2}{3\pi m v_0 c}\frac{v_0^2}{2d} = \frac{\mu_0 q^2 v_0}{6\pi m c d} = \frac{(4\pi\times10^{-7})(1.6\times10^{-19})^2(10^5)}{6\pi(9.11\times10^{-31})(3\times10^8)(3\times10^{-9})} = \boxed{2\times10^{-10}.}$$

So radiative losses due to collisions in an ordinary wire are negligible.

Problem 11.14

$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2} = ma = m\frac{v^2}{r} \Rightarrow v = \sqrt{\frac{1}{4\pi\epsilon_0}\frac{q^2}{mr}}$. At the beginning ($r_0 = 0.5\,\text{Å}$),

$$\frac{v}{c} = \left[\frac{(1.6\times10^{-19})^2}{4\pi(8.85\times10^{-12})(9.11\times10^{-31})(5\times10^{-11})}\right]^{-1/2}\frac{1}{3\times10^8} = 0.0075,$$

and when the radius is one hundredth of this v/c is only 10 times greater (0.075), so for *most* of the trip the velocity is safely nonrelativistic.

¿From the Larmor formula, $P = \frac{\mu_0 q^2}{6\pi c}\left(\frac{v^2}{r}\right)^2 = \frac{\mu_0 q^2}{6\pi c}\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{mr^2}\right)^2$ (since $a = v^2/r$), and $P = -dU/dt$, where U is the (total) energy of the electron:

$$U = U_{\text{kin}} + U_{\text{pot}} = \frac{1}{2}mv^2 - \frac{1}{4\pi\epsilon_0}\frac{q^2}{r} = \frac{1}{2}\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{r}\right) - \frac{1}{4\pi\epsilon_0}\frac{q^2}{r} = -\frac{1}{8\pi\epsilon_0}\frac{q^2}{r}.$$

So $-\frac{dU}{dt} = -\frac{1}{8\pi\epsilon_0}\frac{q^2}{r^2}\frac{dr}{dt} = P = \frac{q^2}{6\pi\epsilon_0 c^3}\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{mr^2}\right)^2$, and hence $\frac{dr}{dt} = -\frac{1}{3c}\left(\frac{q^2}{2\pi\epsilon_0 mc}\right)^2\frac{1}{r^2}$, or

$$dt = -3c\left(\frac{2\pi\epsilon_0 mc}{q^2}\right)^2 r^2\,dr \Rightarrow t = -3c\left(\frac{2\pi\epsilon_0 mc}{q^2}\right)^2\int_{r_0}^{0} r^2\,dr = \boxed{c\left(\frac{2\pi\epsilon_0 mc}{q^2}\right)^2 r_0^3}$$

$$= (3\times10^8)\left[\frac{2\pi(8.85\times10^{-12})(9.11\times10^{-31})(3\times10^8)}{(1.6\times10^{-19})^2}\right]^2(5\times10^{-11})^3 = \boxed{1.3\times10^{-11}\text{ s.}}\ \text{(Not very long!)}$$

Problem 11.15

According to Eq. 11.74, the maximum occurs at $\frac{d}{d\theta}\left[\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right] = 0$. Thus

$\frac{2\sin\theta\cos\theta}{(1-\beta\cos\theta)5} - \frac{5\sin^2\theta(\beta\sin\theta)}{(1-\beta\cos\theta)^6} = 0 \Rightarrow 2\cos\theta(1-\beta\cos\theta) = 5\beta\sin^2\theta = 5\beta(1-\cos^2\theta)$;
$2\cos\theta - 2\beta\cos^2\theta = 5\beta - 5\beta\cos^2\theta$, or $3\beta\cos^2\theta + 2\cos\theta - 5\beta = 0$. So
$\cos\theta = \frac{-2\pm\sqrt{4+60\beta^2}}{6\beta} = \frac{1}{3\beta}\left(\pm\sqrt{1+15\beta^2}-1\right)$. We want the plus sign, since $\theta_m \to 90°(\cos\theta_m = 0)$ when $\beta \to 0$ (Fig. 11.12): $\boxed{\theta_{\max} = \cos^{-1}\left(\frac{\sqrt{1+15\beta^2}-1}{3\beta}\right).}$

For $v \approx c$, $\beta \approx 1$; write $\beta = 1-\epsilon$ (where $\epsilon \ll 1$), and expand to first order in ϵ:

$$\begin{aligned}\left(\frac{\sqrt{1+15\beta^2}-1}{3\beta}\right) &= \frac{1}{3(1-\epsilon)}\left[\sqrt{1+15(1-\epsilon)^2}-1\right] \cong \frac{1}{3}(1+\epsilon)\left[\sqrt{1+15(1-2\epsilon)}-1\right]\\ &= \frac{1}{3}(1+\epsilon)\left[\sqrt{16-30\epsilon}-1\right] = \frac{1}{3}(1+\epsilon)\left[4\sqrt{1-(15\epsilon/8)}-1\right] = \frac{1}{3}(1+\epsilon)\left[4\left(1-\frac{15}{16}\epsilon\right)-1\right]\\ &= \frac{1}{3}(1+\epsilon)\left(3-\frac{15}{4}\epsilon\right) = (1+\epsilon)(1-\frac{5}{4}\epsilon) \cong 1+\epsilon-\frac{5}{4}\epsilon = 1-\frac{1}{4}\epsilon.\end{aligned}$$

Evidently $\theta_{\max} \approx 0$, so $\cos\theta_{\max} \cong 1 - \frac{1}{2}\theta_{\max}^2 = 1 - \frac{1}{4}\epsilon \Rightarrow \theta_{\max}^2 = \frac{1}{2}\epsilon$, or $\theta_{\max} \cong \sqrt{\epsilon/2} = \boxed{\sqrt{(1-\beta)/2}.}$

Let $f \equiv \dfrac{(dP/d\Omega|_{\theta_m})_{\text{ur}}}{(dP/d\Omega|_{\theta_m})_{\text{rest}}} = \left[\dfrac{\sin^2\theta_{\max}}{(1-\beta\cos\theta_{\max})^5}\right]_{\text{ur}}$. Now $\sin^2\theta_{\max} \cong \epsilon/2$, and $(1-\beta\cos\theta_{\max}) \cong 1-(1-\epsilon)(1-\frac{1}{4}\epsilon) \cong 1-(1-\epsilon-\frac{1}{4}\epsilon) = \frac{5}{4}\epsilon$. So $f = \dfrac{\epsilon/2}{(5\epsilon/4)^5} = \left(\dfrac{4}{5}\right)^5 \dfrac{1}{2\epsilon^4}$. But $\gamma = \dfrac{1}{\sqrt{1-\beta^2}} = \dfrac{1}{\sqrt{1-(1-\epsilon)^2}} \cong \dfrac{1}{\sqrt{1-(1-2\epsilon)}} = \dfrac{1}{\sqrt{2\epsilon}} \Rightarrow \epsilon = \dfrac{1}{2\gamma^2}$. Therefore

$$f = \left(\frac{4}{5}\right)^5 \frac{1}{2}(2\gamma^2)^4 = \boxed{\frac{1}{4}\left(\frac{8}{5}\right)^5 \gamma^8 = 2.62\gamma^8.}$$

Problem 11.16

Equation 11.72 says $\dfrac{dP}{d\Omega} = \dfrac{q^2}{16\pi^2\epsilon_0}\dfrac{|\hat{\boldsymbol{\mathscr{r}}}\times(\mathbf{u}\times\mathbf{a})|^2}{(\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{u})^5}$. Let $\beta \equiv v/c$.

$\mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v} = c\hat{\boldsymbol{\mathscr{r}}} - v\hat{\mathbf{z}} \Rightarrow \hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{u} = c - v(\hat{\boldsymbol{\mathscr{r}}}\cdot\hat{\mathbf{z}}) = c - v\cos\theta = c\left(1-\dfrac{v}{c}\cos\theta\right) = c(1-\beta\cos\theta)$;

$\mathbf{a}\cdot\mathbf{u} = ac(\hat{\mathbf{x}}\cdot\hat{\boldsymbol{\mathscr{r}}}) - av(\hat{\mathbf{x}}\cdot\hat{\mathbf{z}}) = ac\sin\theta\cos\phi$; $\quad u^2 = \mathbf{u}\cdot\mathbf{u} = c^2 - 2cv(\hat{\boldsymbol{\mathscr{r}}}\cdot\hat{\mathbf{z}}) + v^2 = c^2 + v^2 - 2cv\cos\theta$.

$$\begin{aligned}
\hat{\boldsymbol{\mathscr{r}}}\times(\mathbf{u}\times\mathbf{a}) &= (\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a})\mathbf{u} - (\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{u})\mathbf{a};\\
|\hat{\boldsymbol{\mathscr{r}}}\times(\mathbf{u}\times\mathbf{a})|^2 &= (\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a})^2u^2 - 2(\mathbf{u}\cdot\mathbf{a})(\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a})(\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{u}) + (\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{u})^2a^2\\
&= (c^2+v^2-2cv\cos\theta)(a\sin\theta\cos\phi)^2 - 2(ac\sin\theta\cos\phi)(a\sin\theta\cos\phi)(c-v\cos\theta) + a^2c^2(1-\beta\cos\theta)^2\\
&= a^2\left[c^2(1-\beta\cos\theta)^2 + (\sin^2\theta\cos^2\phi)(c^2+v^2-2cv\cos\theta-2c^2+2cv\cos\theta\right]\\
&= a^2c^2\left[(1-\beta\cos\theta)^2 - (1-\beta^2)(\sin\theta\cos\phi)^2\right].\\
\frac{dP}{d\Omega} &= \boxed{\frac{\mu_0q^2a^2}{16\pi^2c}\frac{\left[(1-\beta\cos\theta)^2-(1-\beta^2)\sin^2\theta\cos^2\phi\right]}{(1-\beta\cos\theta)^5}.}
\end{aligned}$$

The total power radiated (in all directions) is:

$$P = \int\frac{dP}{d\Omega}\,d\Omega = \int\frac{dP}{d\Omega}\sin\theta\,d\theta\,d\phi = \frac{\mu_0q^2a^2}{16\pi^2c}\int\int\frac{\left[(1-\beta\cos\theta)^2-(1-\beta^2)\sin^2\theta\cos^2\phi\right]}{(1-\beta\cos\theta)^5}\sin\theta\,d\theta\,d\phi.$$

But $\displaystyle\int_0^{2\pi}d\phi = 2\pi$ and $\displaystyle\int_0^{2\pi}\cos^2\phi\,d\phi = \pi$.

$$= \frac{\mu_0q^2a^2}{16\pi^2c}\pi\int_0^{\pi}\frac{\left[2(1-\beta\cos\theta)^2-(1-\beta^2)\sin^2\theta\right]}{(1-\beta\cos\theta)^5}\sin\theta\,d\theta.$$

Let $w \equiv (1-\beta\cos\theta)$. Then $(1-w)/\beta = \cos\theta$; $\quad \sin^2\theta = \left[\beta^2-(1-w)^2\right]/\beta^2$, and the numerator becomes

$$\begin{aligned}
2w^2 - \frac{(1-\beta^2)}{\beta^2}(\beta^2-1+2w-w^2) &= \frac{1}{\beta^2}\left[2w^2\beta^2 + (1-\beta^2)^2 - 2(1-\beta^2)w + w^2(1-\beta^2)\right]\\
&= \frac{1}{\beta^2}\left[(1-\beta^2)^2 - 2(1-\beta^2)w + (1+\beta^2)w^2\right];
\end{aligned}$$

$dw = \beta\sin\theta\,d\theta \Rightarrow \sin\theta\,d\theta = \frac{1}{\beta}\,dw.$ When $\theta = 0,\ w = (1-\beta)$; when $\theta = \pi,\ w = (1+\beta)$.

$$\begin{aligned} P &= \frac{\mu_0 q^2 a^2}{16\pi c}\frac{1}{\beta^3}\int_{(1-\beta)}^{(1+\beta)} \frac{1}{w^5}\left[(1-\beta^2)^2 - 2(1-\beta^2)w + (1+\beta^2)w^2\right]dw. \quad \text{The integral is} \\ \text{Int} &= (1-\beta^2)^2\int \frac{1}{w^5}\,dw - 2(1-\beta^2)\int \frac{1}{w^4}\,dw + (1+\beta^2)\int \frac{1}{w^3}\,dw \\ &= \left[(1-\beta^2)^2\left(-\frac{1}{4w^4}\right) - 2(1-\beta^2)\left(-\frac{1}{3w^3}\right) + (1+\beta^2)\left(-\frac{1}{2w^2}\right)\right]\Bigg|_{1-\beta}^{1+\beta}. \end{aligned}$$

$$\begin{aligned} \frac{1}{w^2}\Big|_{1-\beta}^{1+\beta} &= \frac{1}{(1+\beta)^2} - \frac{1}{(1-\beta)^2} = \frac{(1-2\beta+\beta^2)-(1+2\beta+\beta^2)}{(1+\beta)^2(1-\beta)^2} = -\frac{4\beta}{(1-\beta^2)^2}. \\ \frac{1}{w^3}\Big|_{1-\beta}^{1+\beta} &= \frac{1}{(1+\beta)^3} - \frac{1}{(1-\beta)^3} = \frac{(1-3\beta+3\beta^2-\beta^3)-(1+3\beta+3\beta^2+\beta^3)}{(1+\beta)^3(1-\beta)^3} = -\frac{2\beta(3+\beta^2)}{(1-\beta^2)^3}. \\ \frac{1}{w^4}\Big|_{1-\beta}^{1+\beta} &= \frac{1}{(1+\beta)^4} - \frac{1}{(1-\beta)^4} = \frac{(1-4\beta+6\beta^2-4\beta^3+\beta^4)-(1+4\beta+6\beta^2+4\beta^3+\beta^4)}{(1+\beta)^4(1-\beta)^4} = -\frac{8\beta(1+\beta^2)}{(1-\beta^2)^4}. \end{aligned}$$

$$\begin{aligned} \text{Int} &= (1-\beta^2)^2\left(-\frac{1}{4}\right)\frac{-8\beta(1+\beta^2)}{(1-\beta^2)^4} - 2(1-\beta^2)\left(-\frac{1}{3}\right)\frac{-2\beta(3+\beta^2)}{(1-\beta^2)^3} + (1+\beta^2)\left(-\frac{1}{2}\right)\frac{-4\beta}{(1-\beta^2)^2} \\ &= \frac{2\beta}{(1-\beta^2)^2}\left[(1+\beta^2) - \frac{2}{3}(3+\beta^2) + (1+\beta^2)\right] = \frac{8}{3}\frac{\beta^3}{(1-\beta^2)^2}. \\ P &= \frac{\mu_0 q^2 a^2}{16\pi c}\frac{1}{\beta^3}\frac{8}{3}\frac{\beta^3}{(1-\beta^2)^2} = \boxed{\frac{\mu_0 q^2 a^2 \gamma^4}{6\pi c},} \quad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}}. \end{aligned}$$

Is this consistent with the Liénard formula (Eq. 11.73)? Here $\mathbf{v}\times\mathbf{a} = va(\hat{\mathbf{z}}\times\hat{\mathbf{x}}) = va\,\hat{\mathbf{y}}$, so $a^2 - \left(\frac{\mathbf{v}}{c}\times\mathbf{a}\right)^2 = a^2\left(1-\frac{v^2}{c^2}\right) = (1-\beta^2)a^2 = \frac{1}{\gamma^2}a^2$, so the Liénard formula says $P = \frac{\mu_0 q^2\gamma^6}{6\pi c}\frac{a^2}{\gamma^2}$. ✓

Problem 11.17

(a) To counteract the radiation reaction (Eq. 11.80), you must exert a force $\mathbf{F}_e = -\frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}$. *For circular motion,* $\mathbf{r}(t) = R\left[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}\right]$, $\mathbf{v}(t) = \dot{\mathbf{r}} = R\omega\left[-\sin(\omega t)\,\hat{\mathbf{x}} + \cos(\omega t)\,\hat{\mathbf{y}}\right]$; $\mathbf{a}(t) = \dot{\mathbf{v}} = -R\omega^2\left[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}\right] = -\omega^2\mathbf{r}$; $\dot{\mathbf{a}} = -\omega^2\dot{\mathbf{r}} = -\omega^2\mathbf{v}$. So $\boxed{\mathbf{F}_e = \frac{\mu_0 q^2}{6\pi c}\omega^2\mathbf{v}.}$ $\boxed{P_e = \mathbf{F}_e\cdot\mathbf{v} = \frac{\mu_0 q^2}{6\pi c}\omega^2 v^2.}$ This is the power *you* must supply.

Meanwhile, the power *radiated* is (Eq. 11.70) $P_{\text{rad}} = \frac{\mu_0 q^2 a^2}{6\pi c}$, and $a^2 = \omega^4 r^2 = \omega^4 R^2 = \omega^2 v^2$, so $P_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\omega^2 v^2$, and the two expressions agree.

(b) *For simple harmonic motion,* $\mathbf{r}(t) = A\cos(\omega t)\,\hat{\mathbf{z}}$; $\mathbf{v} = \dot{\mathbf{r}} = -A\omega\sin(\omega t)\,\hat{\mathbf{z}}$; $\mathbf{a} = \dot{\mathbf{v}} = -A\omega^2\cos(\omega t)\,\hat{\mathbf{z}} = -\omega^2\mathbf{r}$; $\dot{\mathbf{a}} = -\omega^2\dot{\mathbf{r}} = -\omega^2\mathbf{v}$. So $\boxed{\mathbf{F}_e = \frac{\mu_0 q^2}{6\pi c}\omega^2\mathbf{v};\ P_e = \frac{\mu_0 q^2}{6\pi c}\omega^2 v^2.}$ But this time $a^2 = \omega^4 r^2 = \omega^4 A^2\cos^2(\omega t)$, whereas $\omega^2 v^2 = \omega^4 A^2 \sin^2(\omega t)$, so

$$P_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\omega^4 A^2 \cos^2(\omega t) \neq P_e = \frac{\mu_0 q^2}{6\pi c}\omega^4 A^2 \sin^2(\omega t);$$

the power you deliver is *not* equal to the power radiated. However, since the time *averages* of $\sin^2(\omega t)$ and $\cos^2(\omega t)$ are equal (to wit: 1/2), *over a full cycle* the energy radiated is the same as the energy input. (In the mean time energy is evidently being stored temporarily in the nearby fields.)

(c) *In free fall,* $\mathbf{v}(t) = \frac{1}{2}gt^2\,\hat{\mathbf{y}}$; $\mathbf{v} = gt\,\hat{\mathbf{y}}$; $\mathbf{a} = g\,\hat{\mathbf{y}}$; $\dot{\mathbf{a}} = 0$. So $\boxed{\mathbf{F}_e = 0;}$ the radiation reaction is zero, and hence $\boxed{P_e = 0.}$ But there *is* radiation: $\boxed{P_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}g^2.}$ Evidently energy is being continuously extracted from the nearby fields. This paradox persists even in the *exact* solution (where we do *not* assume $v \ll c$, as in the Larmor formula and the Abraham-Lorentz formula)—see Prob. 11.31.

Problem 11.18

(a) $\gamma = \omega^2\tau$, and $\tau = 6 \times 10^{-24}$ s (for electrons). Is $\gamma \ll \omega$ (i.e. is $\tau \ll 1/\omega$)? If ω is in the optical region, $\omega = 2\pi\nu = 2\pi(5 \times 10^{14}) = 3 \times 10^{15}$; $1/\omega = (1/3) \times 10^{-15} = 3 \times 10^{-16}$, which is much greater than τ, so the damping is indeed "small". ✓

(b) Problem 9.24 gave $\Delta\omega \cong \gamma = \omega_0^2\tau = [2\pi(7 \times 10^{15})]^2(6 \times 10^{-24}) = \boxed{1 \times 10^{10}\,\text{rad/s.}}$ Since we're in the region of $\omega_0 \approx 4 \times 10^{16}$ rad/s, the width of the anomalous dispersion zone is *very* narrow.

Problem 11.19

(a) $a = \tau\dot{a} + \frac{F}{m} \Rightarrow \frac{dv}{dt} = \tau\frac{da}{dt} + \frac{F}{m} \Rightarrow \int \frac{dv}{dt}\,dt = \tau\int\frac{da}{dt}\,dt + \frac{1}{m}\int F\,dt.$

$[v(t_0+\epsilon) - v(t_0-\epsilon)] = \tau\,[a(t_0+\epsilon) - a(t_0-\epsilon)] + \frac{2\epsilon}{m}F_{\text{ave}}$, where F_{ave} is the average force during the interval. But v is continuous, so as long as F is not a delta function, we are left (in the limit $\epsilon \to 0$) with $[a(t_0+\epsilon) - a(t_0-\epsilon)] = 0$. Thus a, too, is continuous. qed

(b) (i) $a = \tau\dot{a} = \tau\frac{da}{dt} \Rightarrow \frac{da}{a} = \frac{1}{\tau}\,dt \Rightarrow \int\frac{da}{a} = \frac{1}{\tau}\int dt \Rightarrow \ln a = \frac{t}{\tau} + \text{constant} \Rightarrow \boxed{a(t) = Ae^{t/\tau},}$ where A is a constant.

(ii) $a = \tau\dot{a} + \frac{F}{m} \Rightarrow \tau\frac{da}{dt} = a - \frac{F}{m} \Rightarrow \frac{da}{a - F/m} = \frac{1}{\tau}\,dt \Rightarrow \ln(a - F/m) = \frac{t}{\tau} + \text{constant} \Rightarrow a - \frac{F}{m} = Be^{t/\tau} \Rightarrow \boxed{a(t) = \frac{F}{m} + Be^{t/\tau},}$ where B is some other constant.

(iii) Same as (i): $\boxed{a(t) = Ce^{t/\tau},}$ where C is a third constant.

(c) At $t = 0$, $A = F/m + B$; at $t = T$, $F/m + Be^{T/\tau} = Ce^{T/\tau} \Rightarrow C = (F/m)e^{-T/\tau} + B$. So

$$\boxed{a(t) = \begin{cases} [(F/m) + B]\,e^{t/\tau}, & t \le 0; \\ \left[(F/m) + Be^{t/\tau}\right], & 0 \le t \le T; \\ \left[(F/m)e^{-T/\tau} + B\right]e^{t/\tau}, & t \ge T. \end{cases}}$$

To eliminate the runaway in region (iii), we'd need $B = -(F/m)e^{-T/\tau}$; to avoid preacceleration in region (i), we'd need $B = -(F/m)$. Obviously, we cannot do both at once.

(d) If we choose to eliminate the runaway, then

$$\boxed{a(t) = \begin{cases} (F/m)\left[1 - e^{-T/\tau}\right] e^{t/\tau}, & t \le 0; \\ (F/m)\left[1 - e^{(t-T)/\tau}\right], & 0 \le t \le T; \\ 0, & t \ge T. \end{cases}}$$

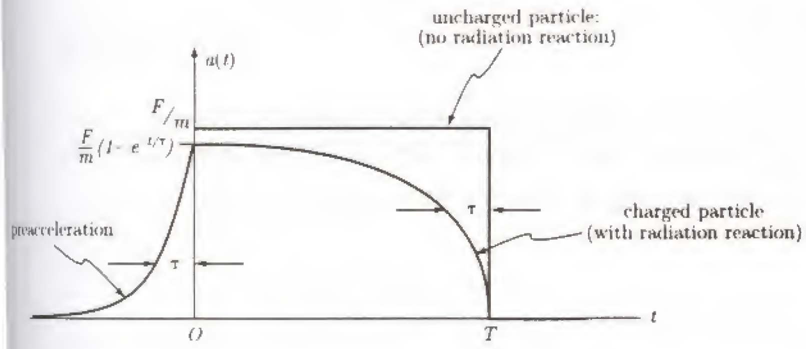
(i) $v = (F/m)\left[1 - e^{-T/\tau}\right] \int e^{t/\tau} dt = (F\tau/m)\left[1 - e^{-T/\tau}\right] e^{t/\tau} + D$, where D is a constant determined by the condition $v(-\infty) = 0 \Rightarrow D = 0$.

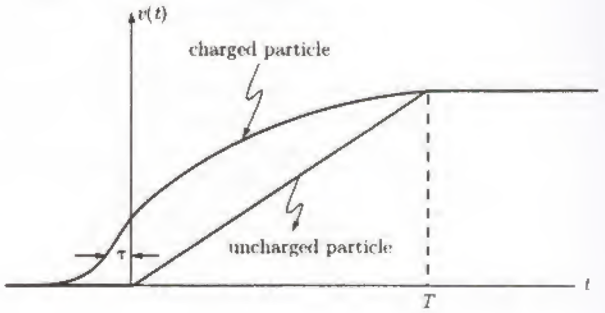
(ii) $v = (F/m)\left[t - \tau e^{(t-T)/\tau}\right] + E$, where E is a constant determined by the continuity of v at $t = 0$: $(F\tau/m)\left[1 - e^{-T/\tau}\right] = (F/m)\left[-\tau e^{-T/\tau}\right] + E \Rightarrow E = (F\tau/m)$.

(iii) v is a constant determined by the continuity of v at $t = T$: $v = (F/m)[T + \tau - \tau] = (F/m)T$.

$$\boxed{v(t) = \begin{cases} (F\tau/m)\left[1 - e^{-T/\tau}\right] e^{t/\tau}, & t \le 0; \\ (F/m)\left[t + \tau - \tau e^{(t-T)/\tau}\right], & 0 \le t \le T; \\ (F/m)T, & t \ge T. \end{cases}}$$

(e)

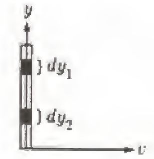




Problem 11.20

(a) From Eq.11.80, $F_{\rm rad}^{\rm end} = \dfrac{\mu_0 (q/2)^2}{6\pi c}\dot{a}$, so $F_{\rm rad} = F_{\rm rad}^{\rm int} + 2F_{\rm rad}^{\rm end} = \dfrac{\mu_0 q^2}{6\pi c}\dot{a}\left[\dfrac{1}{2} + 2\left(\dfrac{1}{4}\right)\right] = \dfrac{\mu_0 q^2}{6\pi c}\dot{a}$. ✓

(b) $F_{\rm rad} = \dfrac{\mu_0}{12\pi c}\dot{a}\displaystyle\int_0^L \left\{\int_0^{y_1} 2\lambda\, dy_2\right\} 2\lambda\, dy_1$. (Running the y_2 integral up to y_1 insures that $y_1 \ge y_2$, so we don't count the same pair twice. Alternatively, run *both* integrals from 0 to L—*intentionally* double-counting—and divide the result by 2.)



$$F_{\text{rad}} = \frac{\mu_0 \dot{a}}{12\pi c}(4\lambda^2)\int_0^L y_1\,dy_1 = \frac{\mu_0\dot{a}}{12\pi c}(4\lambda^2)\frac{L^2}{2} = \frac{\mu_0}{6\pi c}(\lambda L)^2\dot{a} = \frac{\mu_0 q^2}{6\pi c}\dot{a}.\ \checkmark$$

Problem 11.21

(a) This is an oscillating electric dipole, with amplitude $p_0 = qd$ and frequency $\omega = \sqrt{k/m}$. The (averaged) Poynting vector is given by Eq. 11.21: $\langle \mathbf{S}\rangle = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}}$, so the power per unit area of *floor* is

$$\begin{aligned} I_f &= \langle \mathbf{S}\rangle\cdot\hat{\mathbf{z}} = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta\cos\theta}{r^2}. \quad \text{But } \sin\theta = \frac{R}{r},\ \cos\theta = \frac{h}{r},\ \text{and } r^2 = R^2 + h^2. \\ &= \boxed{\left(\frac{\mu_0 q^2 d^2\omega^4}{32\pi^2 c}\right)\frac{R^2 h}{(R^2+h^2)^{5/2}}.} \end{aligned}$$

$$\frac{dI_f}{dR} = 0 \Rightarrow \frac{d}{dR}\left[\frac{R^2}{(R^2+h^2)^{5/2}}\right] = 0 \Rightarrow \frac{2R}{(R^2+h^2)^{5/2}} - \frac{5}{2}\frac{R^2}{(R^2+h^2)^{7/2}}2R = 0 \Rightarrow$$

$(R^2+h^2) - \frac{5}{2}R^2 = 0 \Rightarrow h^2 = \frac{3}{2}R^2 \Rightarrow \boxed{R = \sqrt{2/3}h,}$ for maximum intensity.

(b)

$$\begin{aligned} P &= \int I_f(R)\,da = \int I_f(R)\,2\pi R\,dR = 2\pi\left(\frac{\mu_0 (qd)^2\omega^4}{32\pi^2 c}\right)h\int_0^\infty \frac{R^3}{(R^2+h^2)^{5/2}}\,dR. \quad \text{Let } x \equiv R^2: \\ &\int_0^\infty \frac{R^3}{(R^2+h^2)^{5/2}}\,dR = \frac{1}{2}\int_0^\infty \frac{x}{(x+h^2)^{5/2}}\,dx = \frac{1}{2h}\frac{\Gamma(2)\Gamma(1/2)}{\Gamma(5/2)} = \frac{2}{3h}. \\ &= 2\pi\left(\frac{\mu_0 q^2 d^2\omega^4}{32\pi^2 c}\right)h\frac{2}{3h} = \boxed{\frac{\mu_0 q^2 d^2\omega^4}{24\pi c},} \end{aligned}$$

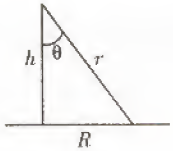
which should be (and *is*) half the *total* radiated power (Eq. 11.22)—the rest hits the ceiling, of course.

(c) The amplitude is $x_0(t)$, so $U = \frac{1}{2}kx_0^2$ is the energy, at time t, and $dU/dt = -2P$ is the power radiated:

$\frac{1}{2}k\frac{d}{dt}(x_0^2) = -\frac{\mu_0\omega^4}{12\pi c}q^2 x_0^2 \Rightarrow \frac{d}{dt}(x_0^2) = -\frac{\mu_0\omega^4 q^2}{6\pi k c}(x_0^2) = -\kappa x_0^2 \Rightarrow x_0^2 = d^2 e^{-\kappa t}$ or $x_0(t) = de^{-\kappa t/2}$.

$$\tau = \frac{2}{\kappa} = \frac{12\pi k c}{\mu_0 q^2 k^2}m^2 = \boxed{\frac{12\pi c m^2}{\mu_0 q^2 k}.}$$

Problem 11.22



(a) From Eq. 11.39, $\langle\mathbf{S}\rangle = \left(\frac{\mu_0 m_0^2\omega^4}{32\pi^2 c^3}\right)\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}}$. Here $\sin\theta = R/r$, $r = \sqrt{R^2+h^2}$, and the total radiated power (Eq. 11.40) is $P = \frac{\mu_0 m_0^2\omega^4}{12\pi c^3}$. So the intensity is $I(R) = \left(\frac{12P}{32\pi}\right)\frac{R^2}{(R^2+h^2)^2} = \boxed{\frac{3P}{8\pi}\frac{R^2}{(R^2+h^2)^2}.}$

(b) The intensity *directly* below the antenna ($R = 0$) would (ideally) have been *zero*. The engineer *should* have measured it at the position of *maximum* intensity:

$$\frac{dI}{dR} = \frac{3P}{8\pi}\left[\frac{2R}{(R^2+h^2)^2} - \frac{2R^2}{(R^2+h^2)^3}2R\right] = \frac{3P}{8\pi}\frac{2R}{(R^2+h^2)^3}\left(R^2+h^2-2R^2\right) = 0 \Rightarrow \boxed{R = h.}$$

At this location the intensity is $I(h) = \dfrac{3P}{8\pi}\dfrac{h^2}{(2h^2)^2} = \boxed{\dfrac{3P}{32\pi h^2}.}$

(c) $I_{\max} = \dfrac{3(35 \times 10^3)}{32\pi(200)^2} = 0.026\,\mathrm{W/m^2} = \boxed{2.6\,\mu\mathrm{W/cm^2}.}$ $\boxed{\text{Yes, KRUD is in compliance.}}$

Problem 11.23

(a) $\mathbf{m}(t) = M\cos\psi\,\hat{\mathbf{z}} + M\sin\psi[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}]$. As in Prob. 11.4, the power radiated will be twice that of an oscillating magnetic dipole with dipole moment of amplitude $m_0 = M\sin\psi$. Therefore (quoting Eq. 11.40): $\boxed{P = \dfrac{\mu_0 M^2\omega^4\sin^2\psi}{6\pi c^3}.}$ (Alternatively, you can get this from the answer to Prob. 11.12.)

(b) From Eq. 5.86, with $r \to R$, $m \to M$, and $\theta = \pi/2$: $B = \dfrac{\mu_0}{4\pi}\dfrac{M}{R^3}$, so

$$\boxed{M = \frac{4\pi R^3}{\mu_0}B} = \frac{4\pi(6.4 \times 10^6)^3(5 \times 10^{-5})}{4\pi \times 10^{-7}} = \boxed{1.3 \times 10^{23}\,\mathrm{A\,m^2}.}$$

(c) $P = \dfrac{(4\pi \times 10^{-7})(1.3 \times 10^{23})^2\sin^2(11^\circ)}{6\pi(3 \times 10^8)^3}\left(\dfrac{2\pi}{24 \times 60 \times 60}\right)^4 = \boxed{4 \times 10^{-5}\,\mathrm{W}}$ (not much).

(d) $P = \dfrac{\mu_0(4\pi R^3 B/\mu_0)^2\omega^4\sin^2\psi}{6\pi c^3} = \dfrac{8\pi}{3\mu_0 c^3}\left(\omega^2 R^3 B\sin\psi\right)^2$. Using the average value (1/2) for $\sin^2\psi$,

$P = \dfrac{8\pi}{3(4\pi \times 10^{-7})(3 \times 10^8)^3}\left[\left(\dfrac{2\pi}{10^{-3}}\right)^2(10^4)^3(10^8)\right]^2\dfrac{1}{2} = \boxed{2 \times 10^{36}\,\mathrm{W}}$ (a lot).

Problem 11.24

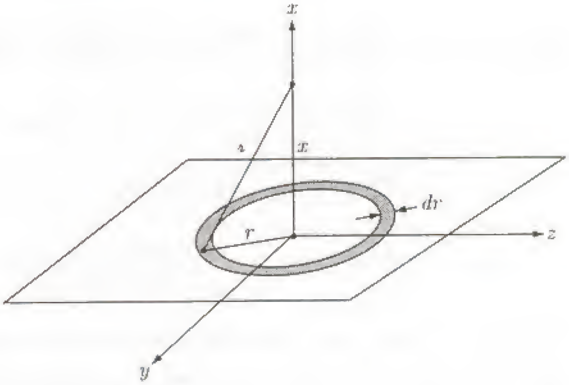
(a) $\mathbf{A}(x,t) = \dfrac{\mu_0}{4\pi}\displaystyle\int\dfrac{\mathbf{K}(t_r)}{\mathscr{r}}\,da$

$= \dfrac{\mu_0\hat{\mathbf{z}}}{4\pi}\displaystyle\int\dfrac{K(t_r)}{\sqrt{r^2+x^2}}2\pi r\,dr$

$= \dfrac{\mu_0\hat{\mathbf{z}}}{2}\displaystyle\int\dfrac{K(t-\sqrt{r^2+x^2}/c)}{\sqrt{r^2+x^2}}r\,dr.$

The maximum r is given by $t - \sqrt{r^2+x^2}/c = 0$;

$r_{\max} = \sqrt{c^2t^2 - x^2}$ (since $K(t) = 0$ for $t < 0$).



(i)

$$\begin{aligned}
\mathbf{A}(x,t) &= \frac{\mu_0 K_0\hat{\mathbf{z}}}{2}\int_0^{r_m}\frac{r}{\sqrt{r^2+x^2}}\,dr = \frac{\mu_0 K_0\hat{\mathbf{z}}}{2}\sqrt{r^2+x^2}\Big|_0^{r_m} = \frac{\mu_0 K_0\hat{\mathbf{z}}}{2}\left(\sqrt{r_m^2-x^2}-x\right) = \frac{\mu_0 K_0(ct-x)}{2}\,\hat{\mathbf{z}}.\\
\mathbf{E}(x,t) &= -\frac{\partial\mathbf{A}}{\partial t} = \boxed{-\frac{\mu_0 K_0 c}{2}\,\hat{\mathbf{z}},}\ \text{for } ct > x, \text{ and } 0, \text{ for } ct < x.\\
\mathbf{B}(x,t) &= \nabla\times\mathbf{A} = -\frac{\partial A_z}{\partial x}\,\hat{\mathbf{y}} = \boxed{\frac{\mu_0 K_0}{2}\,\hat{\mathbf{y}},}\ \text{for } ct > x, \text{ and } 0, \text{ for } ct < x.
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathbf{A}(x,t) &= \frac{\mu_0\alpha\,\hat{\mathbf{z}}}{2}\int_0^{r_m}\frac{(t-\sqrt{r^2+x^2}/c)}{\sqrt{r^2+x^2}}r\,dr = \frac{\mu_0\alpha\hat{\mathbf{z}}}{2}\left[t\int_0^{r_m}\frac{r}{\sqrt{r^2+x^2}}\,dr-\frac{1}{c}\int_0^{r_m}r\,dr\right]\\
&= \frac{\mu_0\alpha\,\hat{\mathbf{z}}}{2}\left[t(ct-x)-\frac{1}{2c}(c^2t^2-x^2)\right] = \frac{\mu_0\alpha\,\hat{\mathbf{z}}}{4c}(x^2-2ctx+c^2t^2) = \frac{\mu_0\alpha(x-ct)^2}{4c}\,\hat{\mathbf{z}}.\\
\mathbf{E}(x,t) &= -\frac{\partial\mathbf{A}}{\partial t} = \boxed{\frac{\mu_0\alpha(x-ct)}{2}\,\hat{\mathbf{z}},}\ \text{for } ct>x,\ \text{and } 0,\ \text{for } ct<x.\\
\mathbf{B}(x,t) &= \boldsymbol{\nabla}\times\mathbf{A} = -\frac{\partial A_z}{\partial x}\,\hat{\mathbf{y}} = \boxed{-\frac{\mu_0\alpha}{2c}(x-ct)\,\hat{\mathbf{y}},}\ \text{for } ct>x,\ \text{and } 0,\ \text{for } ct<x.
\end{aligned}$$

(b) Let $u\equiv\frac{1}{c}\left(\sqrt{r^2+x^2}-x\right)$, so $du=\frac{1}{c}\left[\frac{1}{2}\frac{1}{\sqrt{r^2+x^2}}2r\,dr\right]=\frac{1}{c}\frac{r}{\sqrt{r^2+x^2}}\,dr$, and $t-\frac{\sqrt{r^2+x^2}}{c}=t-\frac{x}{c}-u$, and as $r:0\to\infty$, $u:0\to\infty$. Then $\mathbf{A}(x,t)=\frac{\mu_0 c\,\hat{\mathbf{z}}}{2}\int_0^\infty K\left(t-\frac{x}{c}-u\right)\,du$. qed

$$\begin{aligned}
\mathbf{E}(x,t) &= -\frac{\partial\mathbf{A}}{\partial t} = -\frac{\mu_0 c\,\hat{\mathbf{z}}}{2}\int_0^\infty\frac{\partial}{\partial t}K\left(t-\frac{x}{c}-u\right)\,du.\quad\text{But } \frac{\partial}{\partial t}K\left(t-\frac{x}{c}-u\right)=-\frac{\partial}{\partial u}K\left(t-\frac{x}{c}-u\right).\\
&= \frac{\mu_0 c}{2}\,\hat{\mathbf{z}}\int_0^\infty\frac{\partial}{\partial u}K\left(t-\frac{x}{c}-u\right)\,du = \frac{\mu_0 c}{2}\,\hat{\mathbf{z}}\left[K\left(t-\frac{x}{c}-u\right)\right]\Big|_0^\infty = -\frac{\mu_0 c}{2}\left[K(t-x/c)-K(-\infty)\right]\hat{\mathbf{z}}\\
&= \boxed{-\frac{\mu_0 c}{2}K(t-x/c)\,\hat{\mathbf{z}},}\ [\text{if } K(-\infty)=0].
\end{aligned}$$

Note that (i) and (ii) are consistent with this result. Meanwhile

$$\begin{aligned}
\mathbf{B}(x,t) &= -\frac{\partial A_z}{\partial x}\,\hat{\mathbf{y}} = -\frac{\mu_0 c}{c}\,\hat{\mathbf{y}}\int_0^\infty\frac{\partial}{\partial x}K\left(t-\frac{x}{c}-u\right)\,du.\quad\text{But } \frac{\partial}{\partial x}K\left(t-\frac{x}{c}-u\right)=\frac{1}{c}\frac{\partial}{\partial u}K\left(t-\frac{x}{c}-u\right).\\
&= -\frac{\mu_0}{2}\,\hat{\mathbf{y}}\int_0^\infty\frac{\partial}{\partial u}K\left(t-\frac{x}{c}-u\right)\,du = -\frac{\mu_0}{2}\,\hat{\mathbf{y}}\left[K\left(t-\frac{x}{c}-u\right)\right]\Big|_0^\infty = \frac{\mu_0}{2}\left[K(t-x/c)-K(-\infty)\right]\hat{\mathbf{y}}\\
&= \boxed{\frac{\mu_0}{2}K(t-x/c)\,\hat{\mathbf{y}},}\ [\text{if } K(-\infty)=0].\\
\mathbf{S} &= \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0}\left(\frac{\mu_0 c}{2}\right)\left(\frac{\mu_0}{2}\right)K(t-x/c)\left[-\hat{\mathbf{z}}\times\hat{\mathbf{y}}\right] = \frac{\mu_0 c}{4}\left[K(t-x/c)\right]^2\hat{\mathbf{x}}.
\end{aligned}$$

This is the power per unit area that reaches x at time t; it left the surface at time $(t-x/c)$. Moreover, an equal amount of energy is radiated *downward*, so the total power leaving the surface at time t is $\frac{\mu_0 c}{2}\left[K(t)\right]^2$.

Problem 11.25

$p(t)=2qz(t)$; $\ddot{p}=2q\ddot{z}$; $F=m\ddot{z}=-\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2z)^2}$; $\ddot{z}=-\frac{1}{4\pi\epsilon_0}\frac{q^2}{4mz^2}=-\frac{\mu_0c^2q^2}{16\pi mz^2}$; $\ddot{p}=-\frac{\mu_0c^2q^3}{8\pi mz^2}$.

Using Eq. 11.60, the power radiated is $P=\frac{\mu_0\ddot{p}}{6\pi c}=\frac{\mu_0}{6\pi c}\left(-\frac{\mu_0c^2q^3}{8\pi mz^2}\right)^2=\frac{\mu_0^3c^3q^6}{6(4\pi)^3m^2z^4}=\boxed{\left(\frac{\mu_0cq^2}{4\pi}\right)^3\frac{1}{6m^2z^4}}.$

Problem 11.26

With $\alpha=90°$, Eq. 7.68 gives $\mathbf{E}'=c\mathbf{B}$, $\mathbf{B}'=-\frac{1}{c}\mathbf{E}$, $q'_m=-cq_e$. Use this to "translate" Eqs. 10.65, 10.66, and 11.70:

$$\begin{aligned}
\mathbf{E}' &= c\left(\frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E}\right)=\hat{\boldsymbol{\mathscr{r}}}\times(-c\mathbf{B}')=-c(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{B}').\\
\mathbf{B}' &= -\frac{1}{c}\mathbf{E}=-\frac{1}{c}\frac{q_e}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2-v^2)\mathbf{u}+\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right]\\
&= -\frac{1}{c}\frac{(-q_m'/c)}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2-v^2)\mathbf{u}+\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right]=\frac{\mu_0 q_m'}{4\pi}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2-v^2)\mathbf{u}+\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right].\\
P &= \frac{\mu_0 a^2}{6\pi c}q_e^2=\frac{\mu_0 a^2}{6\pi c}\left(-\frac{1}{c}q_m'\right)^2=\frac{\mu_0 a^2}{6\pi c^3}(q_m')^2.
\end{aligned}$$

Or, dropping the primes,

$$\boxed{\begin{aligned}
\mathbf{B}(\mathbf{r},t) &= \frac{\mu_0 q_m}{4\pi}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2-v^2)\mathbf{u}+\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right].\\
\mathbf{E}(\mathbf{r},t) &= -c(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{B}).\\
P &= \frac{\mu_0 q_m^2 a^2}{6\pi c^3}.
\end{aligned}}$$

Problem 11.27

(a) $W_{\text{ext}}=\int F\,dx=F\int_0^T v(t)\,dt$. From Prob. 11.19, $v(t)=\frac{F}{m}\left[t+\tau-\tau e^{(t-T)/\tau}\right]$. So

$$\begin{aligned}
W_{\text{ext}} &= \frac{F^2}{m}\left[\int_0^T t\,dt+\tau\int_0^T dt-\tau e^{-T/\tau}\int_0^T e^{t/\tau}\,dt\right]=\frac{F^2}{m}\left[\frac{t^2}{2}+\tau t-\tau e^{-T/\tau}\tau e^{t/\tau}\right]\Bigg|_0^T\\
&= \frac{F^2}{m}\left[\frac{1}{2}T^2+\tau T-\tau^2 e^{-T/\tau}\left(e^{T/\tau}-1\right)\right]=\boxed{\frac{F^2}{m}\left(\frac{1}{2}T^2+\tau T-\tau^2+\tau^2 e^{-T/\tau}\right).}
\end{aligned}$$

(b) From Prob. 11.19, the final velocity is $v_f=(F/m)T$, so $W_{\text{kin}}=\frac{1}{2}mv_f^2=\frac{1}{2}m\frac{F^2}{m^2}T^2=\boxed{\frac{F^2T^2}{2m}.}$

(c) $W_{\text{rad}}=\int P\,dt$. According to the Larmor formula, $P=\frac{\mu_0 q^2 a^2}{6\pi c}$, and (again from Prob. 11.19)

$$a(t)=\begin{cases}(F/m)\left[1-e^{-T/\tau}\right]e^{t/\tau}, & (t\le 0);\\ (F/m)\left[1-e^{(t-T)/\tau}\right], & (0\le t\le T).\end{cases}$$

$$\begin{aligned}
W_{\text{rad}} &= \frac{\mu_0 q^2}{6\pi c}\frac{F^2}{m^2}\left\{\left(1-e^{-T/\tau}\right)^2\int_{-\infty}^{0} e^{2t/\tau}\,dt + \int_0^T \left[1-e^{(t-T)/\tau}\right]^2 dt\right\}\\
&= \tau\frac{F^2}{m}\left\{\left(1-e^{-T/\tau}\right)^2\left(\frac{\tau}{2}e^{2t/\tau}\right)\Big|_{-\infty}^{0} + \int_0^T dt - 2e^{-T/\tau}\int_0^T e^{t/\tau}\,dt + e^{-2T/\tau}\int_0^T e^{2t/\tau}\,dt\right\}\\
&= \frac{\tau F^2}{m}\left[\frac{\tau}{2}\left(1-e^{-T/\tau}\right)^2 + T - 2e^{-T/\tau}\left(\tau e^{t/\tau}\right)\Big|_0^T + e^{-2T/\tau}\left(\frac{\tau}{2}e^{2t/\tau}\right)\Big|_0^T\right\}\\
&= \frac{\tau F^2}{m}\left[\frac{\tau}{2}\left(1-2e^{-T/\tau}+e^{-2T/\tau}\right) + T - 2\tau e^{-T/\tau}\left(e^{T/\tau}-1\right) + \frac{\tau}{2}e^{-2T/\tau}\left(e^{2T/\tau}-1\right)\right]\\
&= \frac{\tau F^2}{m}\left[\frac{\tau}{2} - \tau e^{-T/\tau} + \frac{\tau}{2}e^{-2T/\tau} + T - 2\tau + 2\tau e^{-T/\tau} + \frac{\tau}{2} - \frac{\tau}{2}e^{-2T/\tau}\right] = \boxed{\frac{\tau F^2}{m}\left(T-\tau+\tau e^{-T/\tau}\right).}
\end{aligned}$$

Energy conservation requires that the work done by the external force equal the final kinetic energy plus the energy radiated:

$$W_{\text{kin}} + W_{\text{rad}} = \frac{F^2T^2}{2m} + \frac{\tau F^2}{m}\left(T-\tau+\tau e^{-T/\tau}\right) = \frac{F^2}{m}\left(\frac{1}{2}T^2 + \tau T - \tau^2 + \tau^2 e^{-T/\tau}\right) = W_{\text{ext}}.\ \checkmark$$

Problem 11.28

(a) $a = \tau\dot{a} + \dfrac{k}{m}\delta(t) \Rightarrow \displaystyle\int_{-\epsilon}^{\epsilon} a(t)\,dt = v(\epsilon) - v(-\epsilon) = \tau\int_{-\epsilon}^{\epsilon}\frac{da}{dt}\,dt + \frac{k}{m}\int_{-\epsilon}^{\epsilon}\delta(t)\,dt = \tau[a(\epsilon)-a(-\epsilon)] + \frac{k}{m}.$

If the velocity is continuous, so $v(\epsilon) = v(-\epsilon)$, then $\boxed{a(\epsilon) - a(-\epsilon) = -\dfrac{k}{m\tau}.}$

When $t<0$, $a = \tau\dot{a} \Rightarrow a(t) = Ae^{t/\tau}$; when $t>0$, $a = \tau\dot{a} \Rightarrow a(t) = Be^{t/\tau}$; $\quad \Delta a = B - A = -\dfrac{k}{m\tau}$

$\Rightarrow B = A - \dfrac{k}{m\tau}$, so the general solution is $\boxed{a(t) = \begin{cases} Ae^{t/\tau}, & (t<0);\\ \left[A-(k/m\tau)\right]e^{t/\tau}, & (t>0).\end{cases}}$

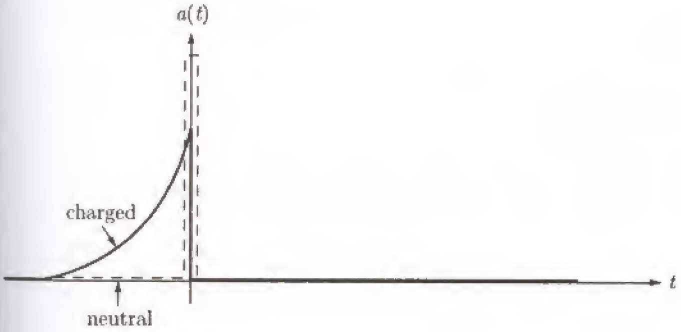
To eliminate the runaway we'd need $A = k/m\tau$; to eliminate preacceleration we'd need $A=0$. Obviously, you can't do both. If you choose to eliminate the runaway, then $\boxed{a(t) = \begin{cases}(k/m\tau)e^{t/\tau}, & (t<0);\\ 0, & (t>0).\end{cases}}$

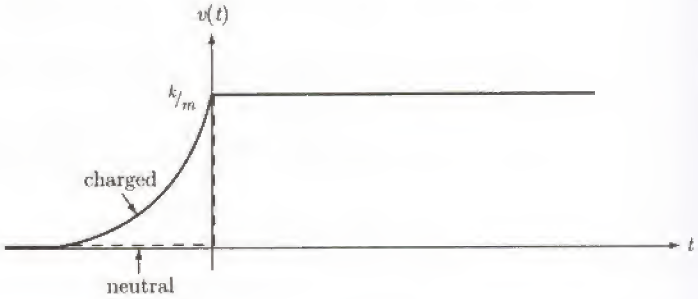
$$v(t) = \int_{-\infty}^{t} a(t)\,dt = \frac{k}{m\tau}\int_{-\infty}^{t} e^{t/\tau}\,dt = \frac{k}{m\tau}\left(\tau e^{t/\tau}\right)\Big|_{-\infty}^{t} = \frac{k}{m}e^{t/\tau}\ \text{(for } t<0\text{)};$$

for $t>0, v(t) = v(0) + \displaystyle\int_0^t a(t)\,dt = v(0) = \frac{k}{m}$. So $\boxed{v(t) = \begin{cases}(k/m)e^{t/\tau}, & (t<0);\\ (k/m), & (t>0).\end{cases}}$

For an *un*charged particle we would have $a(t) = \dfrac{k}{m}\delta(t)$, $v(t) = \displaystyle\int_{-\infty}^{t} a(t)\,dt = \begin{cases}0, & (t<0);\\ (k/m), & (t>0).\end{cases}$

The graphs:





(b)

$$\begin{aligned}
W_{\text{ext}} &= \int F\,dx = \int Fv\,dt = k\int \delta(t)v(t)\,dt = kv(0) = \frac{k^2}{m}.\\
W_{\text{kin}} &= \frac{1}{2}mv_f^2 = \frac{1}{2}m\left(\frac{k}{m}\right)^2 = \frac{k^2}{2m}.\\
W_{\text{ext}} &= \int P_{\text{rad}}\,dt = \frac{\mu_0 q^2}{6\pi c}\int [a(t)]^2\,dt = \tau m\left(\frac{k}{m\tau}\right)^2\int_{-\infty}^{0} e^{2t/\tau}\,dt = \frac{k^2}{m\tau}\left(\frac{\tau}{2}e^{2t/\tau}\right)\Big|_{-\infty}^{0} = \frac{k^2}{m\tau}\frac{\tau}{2} = \frac{k^2}{2m}.
\end{aligned}$$

Clearly, $W_{\text{ext}} = W_{\text{kin}} + W_{\text{rad}}$. ✓

Problem 11.29

Our task is to solve the equation $a = \tau\dot{a} + \dfrac{U_0}{m}[-\delta(x) + \delta(x-L)]$, subject to the boundary conditions

(1) x continuous at $x = 0$ and $x = L$;
(2) v continuous at $x = 0$ and $x = L$;
(3) $\Delta a = \pm U_0/m\tau v$ (plus at $x = 0$, minus at $x = L$).

The third of these follows from integrating the equation of motion:

$$\begin{aligned}
\int \frac{dv}{dt}\,dt &= \tau\int\frac{da}{dt}\,dt + \frac{U_0}{m}\int[-\delta(x) + \delta(x-L)]\,dt,\\
\Delta v &= \tau\Delta a + \frac{U_0}{m}\int[-\delta(x) + \delta(x-L)]\,\frac{dt}{dx}\,dx = 0,\\
\Delta a &= = \frac{U_0}{m\tau}\int\frac{1}{v}[-\delta(x) + \delta(x-L)]\,dx = \pm\frac{U_0}{m\tau v}.
\end{aligned}$$

In each of the three regions the force is zero (it acts only at $x = 0$ and $x = L$), and the general solution is

$$a(t) = Ae^{t/\tau};\quad v(t) = A\tau e^{t/\tau} + B;\quad x(t) = A\tau^2 e^{t/\tau} + Bt + C.$$

(I'll put subscripts on the constants A, B, and C, to distinguish the three regions.)

Region iii ($x > L$): To avoid the runaway we pick $A_3 = 0$; then $a(t) = 0$, $v(t) = B_3$, $x(t) = B_3 t + C_3$. Let the final velocity be v_f ($= B_3$), set the clock so that $t = 0$ when the particle is at $x = 0$, and let T be the time it takes to traverse the barrier, so $x(T) = L = v_f T + C_3$, and hence $C_3 = L - v_f T$. Then

$$\boxed{a(t) = 0;\quad v(t) = v_f,\quad x(t) = L + v_f(t-T),}\quad (t < T).$$

Region ii ($0 < x < L$): $a = A_2 e^{t/\tau}$, $v = A_2\tau e^{t/\tau} + B_2$, $x = A_2\tau^2 e^{t/\tau} + B_2 t + C_2$.

$$(3) \quad \Rightarrow \quad 0 - A_2 e^{T/\tau} = -\frac{U_0}{m\tau v_f} \Rightarrow A_2 = \frac{U_0}{m\tau v_f} e^{-T/\tau}.$$

$$(2) \quad \Rightarrow \quad v_f = A_2 \tau e^{T/\tau} + B_2 = \frac{U_0}{mv_f} + B_2 \Rightarrow B_2 = v_f - \frac{U_0}{mv_f}.$$

$$(1) \quad \Rightarrow \quad L = A_2\tau^2 e^{T/\tau} + B_2 T + C_2 = \frac{U_0 \tau}{mv_f} + v_f T - \frac{U_0 T}{mv_f} + C_2 = v_f T + \frac{U_0}{mv_f}(\tau - T) + C_2 \Rightarrow$$
$$C_2 = L - v_f T + \frac{U_0}{mv_f}(T - \tau).$$

$$\boxed{\begin{aligned} a(t) &= \frac{U_0}{m\tau v_f} e^{(t-T)/\tau}; \\ v(t) &= v_f + \frac{U_0}{mv_f}\left[e^{(t-T)/\tau} - 1\right]; \\ x(t) &= L + v_f(t-T) + \frac{U_0}{mv_f}\left[\tau e^{(t-T)/\tau} - t + T - \tau\right]; \end{aligned}} \quad (0 < t < T).$$

[*Note:* if the barrier is sufficiently wide (or high) the particle may turn around before reaching L, but we're interested here in the régime where it *does* tunnel through.]

In particular, for $t = 0$ (when $x = 0$):

$$0 = L - v_f T + \frac{U_0}{mv_f}\left[\tau e^{-T/\tau} + T - \tau\right] \Rightarrow L = v_f T - \frac{U_0}{mv_f}\left[\tau e^{-T/\tau} + T - \tau\right]. \quad \text{qed}$$

Region i ($x < 0$): $a = A_1 e^{t/\tau}$, $v = A_1\tau e^{t/\tau} + B_1$, $x = A_1\tau^2 e^{t/\tau} + B_1 t + C_1$. Let v_i be the incident velocity (at $t \to -\infty$); then $B_1 = v_i$. Condition (3) says

$$\frac{U_0}{m\tau v_f} e^{-T/\tau} - A_1 = \frac{U_0}{m\tau v_0},$$

where v_0 is the speed of the particle as it passes $x = 0$. From the solution in region (ii) it follows that $v_0 = v_f + \frac{U_0}{mv_f}\left(e^{-T/\tau} - 1\right)$. But we can also express it in terms of the solution in region (i): $v_0 = A_1\tau + v_i$. Therefore

$$\begin{aligned} v_i &= v_f + \frac{U_0}{mv_f}\left(e^{-T/\tau} - 1\right) - A_1\tau = v_f + \frac{U_0}{mv_f}\left(e^{-T/\tau} - 1\right) + \frac{U_0}{mv_0} - \frac{U_0}{mv_f} e^{-T/\tau} \\ &= v_f - \frac{U_0}{mv_f} + \frac{U_0}{mv_0} = v_f - \frac{U_0}{mv_f}\left(1 - \frac{v_f}{v_0}\right) = v_f - \frac{U_0}{mv_f}\left\{1 - \frac{v_f}{v_f + (U_0/mv_f)\left[e^{-T/\tau} - 1\right]}\right\} \\ &= v_f - \frac{U_0}{mv_f}\left\{1 - \frac{1}{1 + (U_0/mv_f^2)\left[e^{-T/\tau} - 1\right]}\right\}. \quad \text{qed} \end{aligned}$$

If $\frac{1}{2}mv_f^2 = \frac{1}{2}U_0$, then

$$L = v_f T - v_f\left[\tau e^{-T/\tau} + T - \tau\right] = v_f\left[T - \tau e^{-T/\tau} - T + \tau\right] = \tau v_f\left(1 - e^{-T/\tau}\right);$$

$$v_i = v_f - v_f\left[1-\frac{1}{1+e^{-T/\tau}-1}\right] = v_f\left(1-1+e^{T/\tau}\right) = v_f e^{T/\tau}.$$

Putting these together,

$$\frac{L}{\tau v_f} = 1-e^{-T/\tau} \Rightarrow e^{-T/\tau} = 1-\frac{L}{\tau v_f} \Rightarrow e^{T/\tau} = \frac{1}{1-(L/\tau v_f)} \Rightarrow v_i = \frac{v_f}{1-(L/v_f\tau)}. \quad \text{qed}$$

In particular, for $L = v_f\tau/4$, $v_i = \frac{v_f}{1-1/4} = \frac{4}{3}v_f$, so $\frac{KE_i}{KE_f} = \frac{\frac{1}{2}mv_i^2}{\frac{1}{2}mv_f^2} = \left(\frac{v_i}{v_f}\right)^2 = \frac{16}{9} \Rightarrow$ $KE_i = \frac{16}{9}KE_f = \frac{16}{9}\frac{1}{2}U_0 = \frac{8}{9}U_0$.

Problem 11.30

(a) From Eq. 10.65, $\mathbf{E}_1 = \frac{(q/2)}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2-v^2)\mathbf{u} + (\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{u} - (\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}\right]$. Here $\mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$, $\boldsymbol{\mathscr{r}} = l\,\hat{\mathbf{x}} + d\,\hat{\mathbf{y}}$, $\mathbf{v} = v\,\hat{\mathbf{x}}$, $\mathbf{a} = a\,\hat{\mathbf{x}}$, so $\boldsymbol{\mathscr{r}}\cdot\mathbf{v} = lv$, $\boldsymbol{\mathscr{r}}\cdot\mathbf{a} = la$, $\boldsymbol{\mathscr{r}}\cdot\mathbf{u} = c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v} = c\mathscr{r} - lv$. We want only the x component. Noting that $u_x = (c/\mathscr{r})l - v = (cl - v\mathscr{r})/\mathscr{r}$, we have:

$$\begin{aligned}
E_{1_x} &= \frac{q}{8\pi\epsilon_0}\frac{\mathscr{r}}{(c\mathscr{r}-lv)^3}\left[\frac{1}{\mathscr{r}}(cl-v\mathscr{r})(c^2-v^2+la) - a(c\mathscr{r}-lv)\right] \\
&= \frac{q}{8\pi\epsilon_0}\frac{1}{(c\mathscr{r}-lv)^3}\left[(cl-v\mathscr{r})(c^2-v^2) + cl^2a - v\mathscr{r}la - ac\mathscr{r}^2 + alv\mathscr{r}\right]. \text{ But } \mathscr{r}^2 = l^2+d^2. \\
&= \frac{q}{8\pi\epsilon_0}\frac{1}{(c\mathscr{r}-lv)^3}\left[(cl-v\mathscr{r})(c^2-v^2) - acd^2\right]. \\
\mathbf{F}_{\text{self}} &= \frac{q^2}{8\pi\epsilon_0}\frac{1}{(c\mathscr{r}-lv)^3}\left[(cl-v\mathscr{r})(c^2-v^2) - acd^2\right]\hat{\mathbf{x}}. \quad \text{(This generalizes Eq. 11.90.)}
\end{aligned}$$

Now $x(t) - x(t_r) = l = vT + \frac{1}{2}aT^2 + \frac{1}{6}\dot{a}T^3 + \cdots$, where $T = t - t_r$, and v, a, and $\dot{a}$ are all evaluated at the retarded time t_r.

$$(cT)^2 = \mathscr{r}^2 = l^2 + d^2 = d^2 + (vT + \frac{1}{2}aT^2 + \frac{1}{6}\dot{a}T^3)^2 = d^2 + v^2T^2 + vaT^3 + \frac{1}{3}v\dot{a}T^4 + \frac{1}{4}a^2T^4;$$

$c^2T^2(1-v^2/c^2) = c^2T^2/\gamma^2 = d^2 + vaT^3 + \left(\frac{1}{3}v\dot{a} + \frac{1}{4}a^2\right)T^4$. Solve for T as a power series in d:

$$T = \frac{\gamma d}{c}\left(1 + Ad + Bd^2 + \cdots\right) \Rightarrow \frac{c^2}{\gamma^2}\frac{\gamma^2 d^2}{c^2}\left(1 + 2Ad + 2Bd^2 + A^2d^2\right) = d^2 + va\frac{\gamma^3 d^3}{c^3}(1+3Ad) + \left(\frac{v\dot{a}}{3} + \frac{a^2}{4}\right)\frac{\gamma^4}{c^4}d^4.$$

Comparing like powers of d: $A = \frac{1}{2}va\frac{\gamma^3}{c^3}$; $2B + A^2 = \frac{3va\gamma^3}{c^3}A + \left(\frac{v\dot{a}}{3} + \frac{a^2}{4}\right)\frac{\gamma^4}{c^4}$.

$$\begin{aligned}
2B &= \frac{3va\gamma^3}{c^3}\frac{1}{2}va\frac{\gamma^3}{c^3} - \frac{1}{4}v^2a^2\frac{\gamma^6}{c^6} + \frac{v\dot{a}}{3}\frac{\gamma^4}{c^4} + \frac{a^2\gamma^4}{4c^4} = \frac{v\dot{a}}{3}\frac{\gamma^4}{c^4} + \frac{\gamma^6a^2}{4c^4}\left(\frac{1}{\gamma^2} - \frac{v^2}{c^2}\right) + \frac{3}{2}\frac{v^2a^2\gamma^6}{c^6} \\
&= \frac{\gamma^4}{c^4}\left[\frac{v\dot{a}}{3} + \frac{a^2\gamma^2}{4}\left(1 - \frac{v^2}{c^2} - \frac{v^2}{c^2} + 6\frac{v^2}{c^2}\right)\right] \Rightarrow B = \frac{\gamma^4}{2c^4}\left[\frac{v\dot{a}}{3} + \frac{\gamma^2a^2}{4}\left(1 + 4\frac{v^2}{c^2}\right)\right]. \\
T &= \frac{\gamma d}{c}\left\{1 + \frac{va}{2}\frac{\gamma^3}{c^3}d + \frac{\gamma^4}{2c^4}\left[\frac{v\dot{a}}{3} + \frac{\gamma^2a^2}{4}\left(1 + 4\frac{v^2}{c^2}\right)\right]d^2\right\} + (\ )\,d^4 + \cdots \quad \text{(generalizing Eq. 11.93).}
\end{aligned}$$

$$\begin{aligned}
l &= vT + \frac{1}{2}aT^2 + \frac{1}{6}\dot{a}T^3 + \cdots \\
&= \frac{v\gamma d}{c}\left\{1 + \frac{va}{2}\frac{\gamma^3}{c^3}d + \frac{\gamma^4}{2c^4}\left[\frac{v\dot{a}}{3} + \frac{\gamma^2 a^2}{4}\left(1 + 4\frac{v^2}{c^2}\right)\right]d^2\right\} + \frac{1}{2}a\frac{\gamma^2 d^2}{c^2}\left[1 + va\frac{\gamma^3}{c^3}d\right] + \frac{1}{6}\dot{a}\frac{\gamma^3}{c^3}d^3 \\
&= \left(\frac{v\gamma}{c}\right)d + \frac{a}{2}\frac{\gamma^4}{c^2}\left(1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}\right)d^2 + \left\{\frac{v\gamma}{2c}\frac{\gamma^4}{c^4}\left[\frac{v\dot{a}}{3} + \frac{\gamma^2 a^2}{4}\left(1 + 4\frac{v^2}{c^2}\right)\right] + \frac{1}{2}a\frac{\gamma^2}{c^2}va\frac{\gamma^3}{c^3} + \frac{1}{6}\dot{a}\frac{\gamma^3}{c^3}\right\}d^3 \\
&= \left(\frac{v\gamma}{c}\right)d + \left(\frac{a\gamma^4}{2c^2}\right)d^2 + \frac{\gamma^3}{2c^3}\left[\frac{\dot{a}}{3}\left(1 + \gamma^2\frac{v^2}{c^2}\right) + \frac{v\gamma^4 a^2}{c^2}\left(\frac{1}{4} + \frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}\right)\right]d^3 \\
&= \left(\frac{v\gamma}{c}\right)d + \left(\frac{a\gamma^4}{2c^2}\right)d^2 + \frac{\gamma^5}{2c^3}\left[\frac{\dot{a}}{3} + \frac{5}{4}\frac{v\gamma^2 a^2}{c^2}\right]d^3 + (\,)\,d^4 + \cdots
\end{aligned}$$

$$\begin{aligned}
\mathscr{r} &= cT = \gamma d\left\{1 + \frac{va}{2}\frac{\gamma^3}{c^3}d + \frac{\gamma^4}{2c^4}\left[\frac{v\dot{a}}{3} + \gamma^2 a^2\left(\frac{1}{4} + \frac{v^2}{c^2}\right)\right]d^2\right\} + (\,)\,d^4 + \cdots \\
c\mathscr{r} - lv &= c\gamma d + \frac{va\gamma^4}{2c^2}d^2 + \frac{\gamma^5}{2c^3}\left[\frac{v\dot{a}}{3} + \gamma^2 a^2\left(\frac{1}{4} + \frac{v^2}{c^2}\right)\right]d^3 - \frac{v^2\gamma}{c}d - \frac{av\gamma^4}{2c^2}d^2 - \frac{\gamma^5 v}{2c^3}\left[\frac{\dot{a}}{3} + \frac{5}{4}\frac{v\gamma^2 a^2}{c^2}\right]d^3 + \cdots \\
&= c\gamma d\left(1 - \frac{v^2}{c^2}\right) + \frac{\gamma^5}{2c^3}\left[\frac{v\dot{a}}{3} + \gamma^2 a^2\left(\frac{1}{4} + \frac{v^2}{c^2}\right) - \frac{v\dot{a}}{3} - \frac{5}{4}\frac{v^2\gamma^2 a^2}{c^2}\right]d^3 + \cdots \\
&= \frac{c}{\gamma}d + \frac{\gamma^5 a^2}{8c^3}d^3 + (\,)\,d^4 + \cdots
\end{aligned}$$

$$\begin{aligned}
cl - v\mathscr{r} &= v\gamma d + \frac{a\gamma^4}{2c}d^2 + \frac{\gamma^5}{2c^2}\left(\frac{\dot{a}}{3} + \frac{5}{4}\frac{v\gamma^2 a^2}{c^2}\right)d^3 - v\gamma d - \frac{v^2 a}{2}\frac{\gamma^4}{c^3}d^2 - \frac{v\gamma^5}{2c^4}\left[\frac{v\dot{a}}{3} + \gamma^2 a^2\left(\frac{1}{4} + \frac{v^2}{c^2}\right)\right]d^3 \\
&= \frac{a\gamma^4}{2c}\left(1 - \frac{v^2}{c^2}\right)d^2 + \frac{\gamma^5}{2c^2}\left[\frac{\dot{a}}{3} + \frac{5}{4}\frac{v\gamma^2 a^2}{c^2} - \frac{v^2}{c^2}\frac{\dot{a}}{3} - \frac{v\gamma^2 a^2}{c^2}\left(\frac{1}{4} + \frac{v^2}{c^2}\right)\right]d^3 + (\,)\,d^4 + \cdots \\
&= \left(\frac{a\gamma^2}{2c}\right)d^2 + \frac{\gamma^5}{2c^2}\left[\frac{\dot{a}}{3\gamma^2} + \frac{v\gamma^2 a^2}{c^2}\left(\frac{5}{4} - \frac{1}{4} - \frac{v^2}{c^2}\right)\right]d^3 + (\,)\,d^4 + \cdots \\
&= \left(\frac{a\gamma^2}{2c}\right)d^2 + \frac{\gamma^3}{2c^2}\left(\frac{\dot{a}}{3} + \frac{v\gamma^2 a^2}{c^2}\right)d^3 + (\,)\,d^4 + \cdots \\
(c\mathscr{r} - lv)^{-3} &= \left[\frac{cd}{\gamma}\left(1 + \frac{\gamma^6 a^2}{8c^4}d^2\right)\right]^{-3} = \left(\frac{\gamma}{cd}\right)^3\left(1 - 3\frac{\gamma^6 a^2}{8c^4}d^2\right) + \cdots
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}_{\text{self}} &= \frac{q^2}{8\pi\epsilon_0}\left(\frac{\gamma}{cd}\right)^3\left(1 - 3\frac{\gamma^6 a^2}{8c^4}d^2\right)\left\{\left[\left(\frac{a\gamma^2}{2c}\right)d^2 + \frac{\gamma^3}{2c^2}\left(\frac{\dot{a}}{3} + \frac{v\gamma^2 a^2}{c^2}\right)d^3\right]\frac{c^2}{\gamma^2} - acd^2\right\}\hat{\mathbf{x}} \\
&= \frac{q^2}{8\pi\epsilon_0}\frac{\gamma^3}{c^3 d}\left(1 - \frac{3}{8}\frac{\gamma^6 a^2}{c^4}d^2\right)\left[-\frac{ac}{2} + \frac{\gamma}{2}\left(\frac{\dot{a}}{3} + \frac{v\gamma^2 a^2}{c^2}\right)d\right]\hat{\mathbf{x}} \\
&= \frac{q^2}{8\pi\epsilon_0}\frac{\gamma^3}{c^3 d}\frac{1}{2}\left[-ac + \gamma\left(\frac{\dot{a}}{3} + \frac{v\gamma^2 a^2}{c^2}\right)d + (\,)\,d^2 + \cdots\right]\hat{\mathbf{x}} \\
&= \frac{q^2}{4\pi\epsilon_0}\left[-\gamma^3\frac{a}{4c^2 d} + \frac{\gamma^4}{4c^3}\left(\frac{\dot{a}}{3} + \frac{v\gamma^2 a^2}{c^2}\right) + (\,)\,d + \cdots\right]\hat{\mathbf{x}} \text{ (generalizing Eq. 11.95).}
\end{aligned}$$

Switching to t: $v(t_r) = v(t) + \dot{v}(t)(t_r - t) + \cdots = v(t) - a(t)T = v(t) - a\gamma d/c$. (When multiplied by d, it doesn't matter—to this order—whether we evaluate at t or at t_r.)

$$1-\left[\frac{v(t_r)}{c}\right]^2 = 1-\frac{[v(t)^2-2va\gamma d/c]}{c^2} = \left[1-\frac{v(t)^2}{c^2}\right]\left(1+\frac{2av\gamma^3 d}{c^3}\right), \text{ so}$$

$$\gamma = \left[1-\left(\frac{v(t_r)}{c}\right)^2\right]^{-1/2} = \gamma(t)\left(1-\frac{va\gamma^3}{c^3}d\right);\ a(t_r)=a(t)-T\dot{a}=a(t)-\frac{\dot{a}\gamma}{c}d.$$

Evaluating everything now at time t:

$$\begin{aligned}
\mathbf{F}_{\text{self}} &= \frac{q^2}{4\pi\epsilon_0}\left[-\gamma^3\frac{(1-3va\gamma^3 d/c^3)(a-\dot{a}\gamma d/c)}{4c^2 d}+\frac{\gamma^4}{4c^3}\left(\frac{\dot{a}}{3}+\frac{v\gamma^2 a^2}{c^2}\right)+(\,)\,d^2+\cdots\right]\hat{\mathbf{x}}\\
&= \frac{q^2}{4\pi\epsilon_0}\left[-\frac{\gamma^3 a}{4c^2 d}+\frac{\gamma^3}{4c^2}\left(\frac{\dot{a}\gamma}{c}+3\frac{va^2\gamma^2}{c^3}\right)+\frac{\gamma^4}{4c^3}\left(\frac{\dot{a}}{3}+\frac{v\gamma^2 a^2}{c^2}\right)+(\,)\,d+\cdots\right]\hat{\mathbf{x}}\\
&= \frac{q^2}{4\pi\epsilon_0}\left[-\frac{\gamma^3 a}{4c^2 d}+\frac{\gamma^4}{4c^3}\left(\dot{a}+\frac{\dot{a}}{3}+3\frac{va^2\gamma^2}{c^2}+\frac{v\gamma^2 a^2}{c^2}\right)+(\,)\,d+\cdots\right]\hat{\mathbf{x}}\\
&= \frac{q^2}{4\pi\epsilon_0}\left[-\frac{\gamma^3 a}{4c^2 d}+\frac{\gamma^4}{3c^3}\left(\dot{a}+3\frac{va^2\gamma^2}{c^2}\right)+(\,)\,d+\cdots\right]\hat{\mathbf{x}} \text{ (generalizing Eq. 11.96).}
\end{aligned}$$

The first term is the electromagnetic mass; the radiation reaction itself is the second term: $F_{\text{rad}}^{\text{int}}=\dfrac{\mu_0 q^2}{12\pi c}\gamma^4\left(\dot{a}+3\dfrac{va^2\gamma^2}{c^2}\right)$ (generalizing Eq. 11.99), so the generalization of Eq. 11.100 is

$$\boxed{F_{\text{rad}}=\frac{\mu_0 q^2}{6\pi c}\gamma^4\left(\dot{a}+3\frac{va^2\gamma^2}{c^2}\right).}$$

(b) $F_{\text{rad}}=A\gamma^4\left(\dot{a}+\dfrac{3\gamma^2a^2v}{c^2}\right)$, where $A\equiv\dfrac{\mu_0 q^2}{6\pi c}$. $P=Aa^2\gamma^6$ (Eq. 11.75). What we must show is that

$$\int_{t_1}^{t_2}F_{\text{rad}}v\,dt=-\int_{t_1}^{t_2}P\,dt, \quad\text{or}\quad \int_{t_1}^{t_2}\gamma^4\left(\dot{a}v+3\frac{v^2a^2\gamma^2}{c^2}\right)dt=-\int_{t_1}^{t_2}a^2\gamma^6\,dt$$

(except for boundary terms—see Sect. 11.2.2).

Rewrite the first term: $\displaystyle\int_{t_1}^{t_2}\gamma^4\dot{a}v\,dt=\int_{t_1}^{t_2}(\gamma^4v)\frac{da}{dt}\,dt=\gamma^4va\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{d}{dt}(\gamma^4v)a\,dt.$

Now $\dfrac{d}{dt}(\gamma^4v)=4\gamma^3\dfrac{d\gamma}{dt}v+\gamma^4a$; $\dfrac{d\gamma}{dt}=\dfrac{d}{dt}\left(\dfrac{1}{\sqrt{1-v^2/c^2}}\right)=-\dfrac{1}{2}\dfrac{1}{(1-v^2/c^2)^{3/2}}\left(-\dfrac{2va}{c^2}\right)=\dfrac{va\gamma^3}{c^2}$. So

$$\frac{d}{dt}(\gamma^4v)=4\gamma^3v\frac{va\gamma^3}{c^2}+\gamma^4a=\gamma^6a\left(1-\frac{v^2}{c^2}+4\frac{v^2}{c^2}\right)=\gamma^6a\left(1+3\frac{v^2}{c^2}\right).$$

$$\int_{t_1}^{t_2}\gamma^4\dot{a}v\,dt=\gamma^4va\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\gamma^6a^2\left(1+3\frac{v^2}{c^2}\right)dt, \text{ and hence}$$

$$\int_{t_1}^{t_2}\gamma^4\left(\dot{a}v+\frac{3\gamma^2a^2v^2}{c^2}\right)dt=\gamma^4va\Big|_{t_1}^{t_2}+\int_{t_1}^{t_2}\left[-\gamma^6a^2\left(1+3\frac{v^2}{c^2}\right)+3\gamma^6\frac{a^2v^2}{c^2}\right]dt=\gamma^4va\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\gamma^6a^2\,dt. \quad \text{qed}$$

Problem 11.31

(a) $P=\dfrac{\mu_0q^2a^2\gamma^6}{6\pi c}$ (Eq. 11.75). $w=\sqrt{b^2+c^2t^2}$ (Eq. 10.45); $v=\dot{w}=\dfrac{c^2t}{\sqrt{b^2+c^2t^2}}$;

$$a=\dot{v}=\frac{c^2}{\sqrt{b^2+c^2t^2}}-\frac{c^2t(c^2t)}{(b^2+c^2t^2)^{3/2}}=\frac{c^2}{(b^2+c^2t^2)^{3/2}}(b^2+c^2t^2-c^2t^2)=\frac{b^2c^2}{(b^2+c^2t^2)^{3/2}};$$

$\gamma^2 = \frac{1}{1 - v^2/c^2} = \frac{1}{1 - [c^2t^2/(b^2 + c^2t^2)]} = \frac{b^2 + c^2t^2}{b^2 + c^2t^2 - c^2t^2} = \frac{1}{b^2}\left(b^2 + c^2t^2\right)$. So

$P = \frac{\mu_0 q^2}{6\pi c}\frac{b^4c^4}{(b^2 + c^2t^2)^3}\frac{(b^2 + c^2t^2)^3}{b^6} = \boxed{\frac{q^2 c}{6\pi\epsilon_0 b^2}.}$ $\boxed{\text{Yes, it radiates}}$ (in fact, at a *constant rate*).

(b) $F_{\text{rad}} = \frac{\mu_0 q^2 \gamma^4}{6\pi c}\left(\dot{a} + \frac{3\gamma^2 a^2 v}{c^2}\right)$; $\dot{a} = -\frac{3}{2}\frac{b^2c^2(2c^2t)}{(b^2 + c^2t^2)^{5/2}} = -\frac{3b^2c^4t}{(b^2 + c^2t^2)^{5/2}}$; $\left(\dot{a} + \frac{3\gamma^2 a^2 v}{c^2}\right) =$ $-\frac{3b^2c^4t}{(b^2 + c^2t^2)^{5/2}} + \frac{3}{c^2}\frac{(b^2 + c^2t^2)}{b^2}\frac{b^4c^4}{(b^2 + c^2t^2)^3}\frac{c^2t}{\sqrt{b^2 + c^2t^2}} = 0.$ $\boxed{F_{\text{rad}} = 0.}$ $\boxed{\text{No, the radiation reaction is zero.}}$